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Application of Homotopy Analysis Method to Homoclinic and Heteroclinic Orbits

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ABSTRACT. Nonlinear problems have been studied in various fields, and it is still a hot topic to search for their solutions. This paper focuses on the homotopy analysis method (HAM) to solve homoclinic and heteroclinic orbits arising in nonlinear dynamics. HAM provides an effective way to adjust and control the convergence of the series solution and improves the computational efficiency. Padé approximate is adopted to accelerate the convergence, and the results are much better than those by HAM alone.

1. Introduction

In recent years, various nonlinear problems arise in many fields of science, such as mathematics, physics, chemistry, biology, medicine, economics, engineering and cybernetics. In the process of solving these nonlinear problems, modern mathematics has gradually produced very important methods and theories, including the Melnikov method, the multiple scales method, the homotopy analysis method (HAM) and so on. These methods have become effective theoretical tools for solving the nonlinear problems in the field of science and technology. In this thesis, the homotopy analysis method is used to solve homoclinic orbits and heteroclinic orbits in nonlinear systems. And because of the complexity of objective things, nonlinear approximation is gradually popular in academia. Padé approximation as the one of the most important rational approximation which has been widely attention.

In 1997, Li and McLaughlin [1] combined Melnikov analysis with a geometric singular perturbation theory and a purely geometric argument, the existence of homoclinic orbits was established. In 1998, Camassa et al [2] propose an extension of the Melnikov method which can be used to confirm the existence of homoclinic and

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heteroclinic orbits in a class of near integrable systems. In 2017, Shi [3] used the Melnikov method to consider the heteroclinic bifurcation and chaos of Josephson system. Moreover, the influence of parameters of system on dynamic behavior is investigated by numerical simulation. In 2001, Wang et al [4] researched a class of two-degree-of-freedom systems in the way of the multiple scales method. The existence of periodic solutions of the system is proved, they provided a formula for determining the existence of homoclinic orbit in a system. In 2004, Beyn et al [5] studied a numerical method for approximating heteroclinic orbits using finite orbital sequences for heteroclinic orbit of discrete time dynamical systems. By an example,they proved the approximation method and the error estimate is valid. In 2005, Koltsova [6] discussed a real analytic Hamiltonian system with two degrees of freedom by use of two-parameter unfolding, which having a homoclinic orbit to a saddle-center equilibrium. In 2006, Wilczak [7] provided a new topological tool to prove the existence of Shilnikov homoclinic or heteroclinic solutions. In 2009, Deng et al [8] researched a class of second order discrete Hamiltonian systems without any periodicity assumptions. According to the critical point theory, they obtained some sufficient conditions for the existence of homoclinic orbits. In 2009, Chen et al [9] used the hyperbolic perturbation method to solve homoclinic and heteroclinic solutions of strongly nonlinear autonomous oscillators. By comparison with Runge–Kutta method, they found that the hyperbolic perturbation method is valid. In 2010, Chen et al [10] utilized the hyperbolic Lindstedt-Poincaré method to solve homoclinic and heteroclinic solutions of strongly nonlinear autonomous oscillators.

By comparison with Runge–Kutta method, they found that the hyperbolic perturbation method is effective. In 2011, Deng et al [11] discussed the second order nonlinear p-Laplacian difference equations. They proved the existence of nontrivial homoclinic orbits of this system. In 2013, Shi et al [12] according to the mountain pass theorem and symmetric mountain pass theorem in critical point theory, they proved the existence of at least one or infinitely many homoclinic solutions for the giving p-Laplacian system. In 2015, Huang et al [13][12] utilized the variational method and critical point theory to explore the existence of heteroclinic solutions for the second-order ifurcations of Duffing-harmonic-van de Pol oscillator in the way of modified generalizHamiltonian system. In 2016, Li et al [14] discussed the homoclinic and heteroclinic bed harmonic function Lindstedt–Poincaré method. By comparison with Runge–Kutta method, they found that the presented method is effective. Similarly, there are a large number of studies have been paid to the heteroclinic orbits in nonlinear dynamics. Now, we consider the homotopy analysis method in homoclinic orbits and heteroclinic orbits in nonlinear dynamics.

In [15], Liao put forward a method, this approach does not depend upon any small parameters at all. Therefore, it is valid for solving nonlinear systems. In 2009, Mehdi Ganjiani et al [16] utilized homotopy analysis methods for solving a coupled nonlinear diffusion–reaction equations. The effectiveness of this method has been proved. In 2012, Qian et al [17] they mainly consider the multi-degree-of-freedom (MDOF) nonlinear non-autonomous dynamical systems through extended homotopy analysis methods, which can overcome the foregoing barriers of conventional asymptotic techniques. In 2014, Qian et al [18] based on the homotopy analysis method, they considered the homoclinic orbit in an autonomous buckled thin plate system. By comparison, they found that the homotopy analysis method is an effective technique of analytic approximation for a homoclinic orbit. In 2014, Liu et al [19] discussed the heteroclinic orbits of Michelson system with the homotopy analysis method. By comparison with exact solution, they proved the effectiveness of the homotopy analysis method. In 2017, Farid Tajaddodianfar et al [20] utilized the homotopy analysis methods to research the derivation of analytical solutions for the frequency response of the resonators. Numerical simulations are performed to validate that the analytical results is effective. For the Padé approximate, it is a simple and effective method to obtain rational function approximation from power series. In 2006, Ismail [21] used Padé approximate to receive the closed-form approximate expressions for the moment-generating function. It is clear that Padé approximate is efficacious. In 2009, Kandayan [22] considered the multipoint Padé approximations of the beta function and gained the limit distribution of the zeros of the denominators. In 2010, Kandayan et al [23] studied the multipoint Hermite-Padé approximations of two beta functions generating the Nikishin system. The result is explained by the vector equilibrium problem in logarithmic potential theory. In 2014, Starovoitov [24] utilized the Laplace method to discuss the asymptotic properties of the Hermite integrals and obtained the asymptotic form of the diagonal Hermite-Padé approximations for the system of exponents. In 2016, on the basis of orthogonal function, Qian et al [25] studied the Padé approximation problem under the orthogonal polynomial and orthogonal trigonometric function. And they provided the specific examples to show the valid of the Padé approximation.

In section 1, we give some basic definitions. In section 2, we give three specific examples. In section 3, it is the summary. In section 4, it is the reference.

2. HAM and Homotopy Padé Approximation

2.1. Homotopy Analysis Method

Multiple degrees of freedom nonlinear system is considered as follows:

$$M\ddot{q} + G\dot{q} + Kq = F(\dot{q}, q, t), \tag{1}$$

in which, q is an n-dimensional unknown vector, M is mass coefficient matrix, G is damping coefficient matrix, K is linear coefficient of stiffness matrix, they are $n \times n$ real number matrixes. In addition to, F is the vector value function of $\dot{q}$, q and t.

Following, we use homotopy analysis method to solve the Eq. (1), a nonlinear operator is constituted as

$$N[u(r,t)] = M\frac{\partial^2 u(r,t)}{\partial t^2} + G\frac{\partial u(r,t)}{\partial t} + Ku(r,t) - F\left(\frac{\partial u}{\partial t}, u, t\right), \tag{2}$$

where, $u(r,t)$ is unknown vector value function.

$$u(r,t) = (x_1(t), \dots, x_n(t))^T, \tag{3}$$

$$\frac{\partial u(r,t)}{\partial t} = \left(\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt}\right)^T, \tag{4}$$

$$\frac{\partial u^2(r,t)}{\partial t^2} = \left(\frac{d^2 x_1}{dt^2}, \dots, \frac{d^2 x_n}{dt^2}\right)^T. \tag{5}$$

On the basis of the essential concepts of HAM, the zero order deformation equation is constructed as

$$(1-p)\{L[\Phi(r,t;p) - u_0(r,t)]\} = phH(t)N\big(\Phi(r,t;p)\big). \tag{6}$$

Letting $u_0(r,t)$ is the initial guess solution of the Eq. (1), h is nonzero constant, $H(t)$ is nonzero real function, L is an auxiliary linear operator, $p \in [0,1]$ is embedding parameter.

For $p = 0$, according to the properties of L, we can know that $\Phi(r, t; 0) = u_0(r, t)$ is the solution of Eq. (6), for $p = 1$, $\Phi(r, t; 1) = u(r, t)$ is the solution of Eq. (6). Therefore, according to the continuous dependence of the intial value processing of the solution of the differential equation, we can conclude that as p increases from 0 to 1, the solution $\Phi(r, t; p)$ transform from the initial guess $u_0(r, t)$ to the exact solution $u(r, t)$.

Now, we give the definition of m order deformation derivatives,

$$u_0^{[m]}(r, t) = \left.\frac{\partial^m \Phi(r, t; p)}{\partial p^m}\right|_{p=0}, \tag{7}$$

$$u_m(r, t) = \frac{u_0^{[m]}(r, t)}{m!} = \frac{1}{m!}\left.\frac{\partial^m \Phi(r, t; p)}{\partial p^m}\right|_{p=0}, \tag{8}$$

applying the Taylor expansion theorem of vector value function, we can get that $\Phi(r, t; p)$ can be expanded into a power series with respect to p

$$\Phi(r, t; p) = u_0(r, t) + \sum_{m=1}^{+\infty} \frac{u_0^{[m]}(r, t)}{m!} p^m, \tag{9}$$

$$\Phi(r, t; p) = u_0(r, t) + \sum_{m=1}^{+\infty} u_m(r, t)\, p^m. \tag{10}$$

Proved that the initial guess solution, linear auxiliary operator, auxiliary parameter h and auxiliary function $H(t)$ are reasonable chosen, the series expansion of equation (10) convergence at $p = 1$, hence

$$u(r, t) = u_0(r, t) + \sum_{m=1}^{+\infty} u_m(r, t). \tag{11}$$

For simplicity, we define the u_m as the following form

$$u_m = \{u_0(r, t), u_1(r, t), \ldots, u_m(r, t)\}, \tag{12}$$

Differentiating the zeroth-order deformation Eq. (6) m times with regard to p, then dividing the equation by $m!$ and setting $p = 0$, we can obtain that

$$L[u_m(r, t) - \chi_m u_{m-1}(r, t)] = hH(t) R_m(u_{m-1}(r, t)), \tag{13}$$

in which

$$\chi_m = \begin{cases} 0, m \le 1 \\ 1, m > 1 \end{cases}, \tag{14}$$

as well as

$$R_m(u_{m-1}(r, t)) = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1} N(\Phi(r, t; p))}{\partial p^{m-1}}\right|_{p=0}, \tag{15}$$

then, we can further obtain that

$$R_m(u_{m-1}(r,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}} N\left[\sum_{n=0}^{+\infty} u_n(r,t)\, p^n\right]\Bigg|_{p=0}. \tag{16}$$

Therefore, by solving the solution of linear higher order deformation Eq. (13), we can get $u_1(r,t),\dots,u_m(r,t)$ with using mathematical software such as Mathematica and Maple.

2.2 Padé Approximation based on orthogonal basis functions

Letting:

$$f(x) = \sum_{j=0}^{+\infty} a_j x^j, a_0 \neq 0, \tag{17}$$

$$(u(x), v(x)) = \int_a^b \left[m u(x) v(x) + n u^{(i)}(x) v^{(i)}(x)\right] dx. \tag{18}$$

According to the definition of the inner product, we can receive the orthogonal basis function$\{\varphi_i(x) | i = 1,2,3,\dots\}$. For example,

$$(u_i(x), v_j(x)) = \int_0^1 \left[u_i(x) v_j(x) + u_i'(x) v_j'(x)\right] dx, \tag{19}$$

$$\varphi_1(x) = 1, \varphi_2(x) = x - \frac{1}{2}, \varphi_3(x) = x^2 - x - \frac{1}{6}, \tag{20}$$

$$\varphi_4(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}, \tag{21}$$

$$\varphi_5(x) = x^4 - 2x^3 + \frac{7}{9}x^2 - \frac{2}{7}x + \frac{1}{70}. \tag{22}$$

We call that

$$\frac{\sum_{i=1}^{m} a_i \varphi_i(x)}{1 + \sum_{j=2}^{n} b_j \varphi_j(x)} \tag{23}$$

is $[m,n]$ order Padé approximation of the $f(x)$, if

$$\left\langle f(x)\left(1 + \sum_{j=2}^{n} b_j \varphi_j(x)\right) - \sum_{i=1}^{n} a_i \varphi_j(x), \varphi_k(x)\right\rangle = 0, \tag{24}$$

where $k = 1,2,3,\dots$

By using the homotopy analysis method to solve the nonlinear system, we can obtain the homopoty analysis solution of the system $u(r,t)$. So, according to the above process, we can obtain the $[m,n]$ order Padé approximation of the homopoty solution $u(r,t)$.

$$\frac{\sum_{i=1}^{m} a_i \varphi_i(x)}{1+\sum_{j=2}^{n} b_j \varphi_j(x)} \approx u(r,t), \tag{25}$$

3. Illustrative Examples and Discussion

In this section, we used some illustrative examples to demonstrate the applicability and accuracy of the HAM for the nonlinear systems. The system is mainly calculated with Mathematica software.

3.1 Example 1

In this section, we apply the HAM for the Mechelson system, which is as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = c^2 - x_2 - {x_1}^2/2 \end{cases}. \tag{26}$$

Letting

$$c = 0.84952, \tag{27}$$

Where

$$x_1(0) = 0, x_2(0) = -\frac{1485}{722}, x_3(0) = 0, \tag{28}$$

the exact homoclinic solutions of the Eq. (26), which satisfied the condition (28) as follow:

$$x_1(t) = \alpha[-9tanh(\beta t) + 11tanh^3(\beta t)], \tag{29a}$$

$$x_2(t) = \alpha[-9\beta sech^2(\beta t) + 33\beta sech^2(\beta t)tanh^2(\beta t)], \tag{29b}$$

$$\begin{aligned} x_3(t) = \alpha[&18\beta^2 sech^2(\beta t)tanh(\beta t) \\ &+ 11(6\beta^2 sech^4(\beta t)\tanh(\beta t) - 6\beta^2 \operatorname{sech}(\beta t)\, tanh^3(\beta t)], \end{aligned} \tag{29c}$$

We select $\{\varphi_i | i = 1,2,3,\dots\}$ as the basis function. According to the principle of solution expression and the principle of coefficient traversal, we choose an initial approximation as below:

$$x_{1,0}(t) = -kx, x_{2,0}(t) = -k, x_{3,0}(t) = 0. \tag{30}$$

We define the linear operator and non-linear operator as follows:

$$L\begin{pmatrix} \varphi_1(t;q) \\ \varphi_2(t;q) \\ \varphi_3(t;q) \end{pmatrix} = \begin{pmatrix} \frac{\partial \varphi_1(t;q)}{\partial t} \\ \frac{\partial \varphi_2(t;q)}{\partial t} \\ \frac{\partial \varphi_3(t;q)}{\partial t} \end{pmatrix}, \tag{31}$$

$$N\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\\ \varphi_3(t;q)\end{pmatrix}=\begin{pmatrix}\dfrac{\partial\varphi_1(t;q)}{\partial t}-\varphi_2(t;q)\\ \dfrac{\partial\varphi_2(t;q)}{\partial t}-\varphi_3(t;q)\\ \dfrac{\partial\varphi_3(t;q)}{\partial t}-c^2+\varphi_2(t;q)+[\varphi_1(t;q)]^2/2\end{pmatrix}. \tag{32}$$

So, we can get the zero-order deformation equation of the system (26):

$$(1-q)L\begin{pmatrix}\varphi_1(t;q)-x_{1,0}(t)\\ \varphi_2(t;q)-x_{2,0}(t)\\ \varphi_3(t;q)-x_{3,0}(t)\end{pmatrix}=qhH(t)N\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\\ \varphi_3(t;q)\end{pmatrix}. \tag{33}$$

Thus, the m-order deformation equation can be shown as

$$L\begin{pmatrix}x_{1,m}(t)-\chi_m x_{1,m-1}(t)\\ x_{2,m}(t)-\chi_m x_{2,m-1}(t)\\ x_{3,m}(t)-\chi_m x_{3,m-1}(t)\end{pmatrix}=hH(t)N\begin{pmatrix}R_{1,m}(x_{1,m-1})\\ R_{2,m}(x_{2,m-1})\\ R_{3,m}(x_{3,m-1})\end{pmatrix}. \tag{34}$$

Supposing

$$H(t)=1, \tag{35}$$

based on the initial conditions and initial values, we can chalk up

$$x_{1,m}(0)=0, x_{2,m}(0)=0, x_{3,m}(0)=0, (m\geq 1), \tag{36}$$

in addition

$$R_{1,m}(x_{1,m-1})=x'_{1,m-1}(t)-x_{2,m-1}(t), \tag{37a}$$

$$R_{2,m}(x_{2,m-1})=x'_{2,m-1}(t)-x_{3,m-1}(t), \tag{37b}$$

$$R_{3,m}(x_{3,m-1})=x'_{3,m-1}(t)+x_{2,m-1}(t)-(1-\chi_m)c^2+\frac{1}{2}\sum_{i=0}^{m-1}x_{1,i}x_{1,m-1-i}. \tag{37c}$$

By solving the Eq. (34), we can obtain

$$x_{1,m}(t)=\chi_m x_{1,m-1}(t)+\hbar\int_0^t R_{1,m}(x_{1,m-1})ds, \tag{38a}$$

$$x_{2,m}(t)=\chi_m x_{2,m-1}(t)+\hbar\int_0^t R_{2,m}(x_{2,m-1})ds, \tag{38b}$$

$$x_{3,m}(t)=\chi_m x_{3,m-1}(t)+\hbar\int_0^t R_{3,m}(x_{3,m-1})ds, \tag{38c}$$

$$x_{1,1}(t)=-\frac{1485t}{722}, x_{2,1}(t)=-\frac{1485}{722}, x_{3,1}(t)=\hbar\left(-\frac{38115t}{13718}+\frac{735075t^3}{1042568}\right). \tag{39}$$

So, we can gain the m-order analytic approximate solution of heteroclinic orbits can be indicated as

$$x(t) = (x_1(t), x_2(t), x_3(t)), \tag{40}$$

where

$$x_1(t) = x_{1,0}(t) + x_{1,1}(t) + x_{1,2}(t) + \cdots + x_{1,m}(t), \tag{41a}$$

$$x_2(t) = x_{2,0}(t) + x_{2,1}(t) + x_{2,2}(t) + \cdots + x_{2,m}(t), \tag{41b}$$

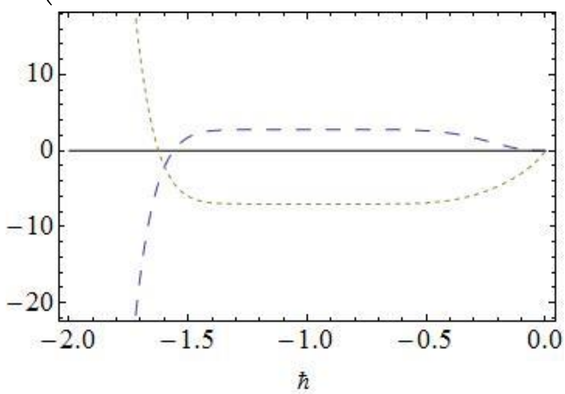
$$x_3(t) = x_{3,0}(t) + x_{3,1}(t) + x_{3,2}(t) + \cdots + x_{3,m}(t). \tag{41c}$$

Then, we can get 4-order analytic approximate solution in the way of HAM as follows:

$$x_1(t) \approx -\frac{1485t}{722} + \frac{1815\hbar^3(2+3\hbar)t^3(-1064+81t^3)}{4170272} + 2\hbar\left(-\frac{12705\hbar^2t^3}{27436} + \frac{147015\hbar^2t^5}{4170272}\right), \tag{42a}$$

$$\begin{aligned} x_2(t) \approx & -\frac{1485}{722} - \frac{5445\hbar^2(1+2\hbar)t^2(-1064+135t^3)}{2085136} + 3\hbar\left(\frac{38115\hbar t^2}{27436} - \frac{735075\hbar t^4}{4170272}\right) \\ & +\hbar\left(\frac{38115\hbar t^2}{27436} + \frac{38115\hbar^2t^2}{6859} + \frac{114345\hbar^3t^2}{27436} - \frac{735075\hbar t^4}{4170272}\right) \\ & +\hbar\left(-\frac{735075\hbar t^4}{4170272} - \frac{2688015\hbar^3t^4}{4170272} + \frac{49005\hbar^3t^6}{4170272}\right) \end{aligned}, \tag{42b}$$

$$\begin{aligned} x_3(t) \approx & \ 4\hbar\left(-\frac{385115t}{13718} + \frac{735075t^3}{1042568}\right) + 3\hbar\left(-\frac{38115\hbar t}{13718} + \frac{735075\hbar t^3}{1042568}\right) - 2\hbar\frac{38115\hbar t}{13718} \\ & +2\hbar\left(-\frac{385115\hbar^2t}{13718} + \frac{735075\hbar t^3}{1042568} + \frac{1217865\hbar^2t^3}{1042568} - \frac{147015\hbar^2t^5}{4170272}\right) - \hbar\frac{38115\hbar t}{13718} \\ & +\hbar\left(-\frac{385115\hbar^2t}{6859} - \frac{38115\hbar^3t}{13718} + \frac{735075\hbar t^3}{1042568} + \frac{1217865\hbar^2t^3}{521284} + \frac{2183445\hbar^3t^3}{1042568}\right) \\ & +\hbar\left(-\frac{147015\hbar^2t^5}{2085136} + \frac{6713685\hbar^3t^5}{79235168} - \frac{218317275\hbar^3t^7}{21076554688}\right) \end{aligned}. \tag{42c}$$



---------$x_1^m(0)\sim\hbar$ ____$x_2^m(0)\sim\hbar$ and ---------$x_3^m(0)\sim\hbar$

Figure 1. $\hbar$ -curves of $x_1^m(0)$ and $x_2^m(0)$ obtained from 10th-order approximation for Eq. (26)

By drawing $\hbar$ -curve in Figure1, we can choose the effective range of parameters. Within the valid range of $\hbar$, the approximation solution $x(t) = (x_1(t), x_2(t), x_3(t))$ are convergent. Letting $\hbar = -1$, the 10-order analytic approximate solution in the way of HAM are

$$x_1(t) = -\frac{1485t}{722} + \frac{12705t^3}{27436} - \frac{243573t^5}{4170272} + \frac{6573809t^7}{1109292352} - \frac{57817309t^9}{108393709824} + \frac{101386197t^{11}}{3203636312576} - \frac{1027487835t^{13}}{1808681529614336}, \tag{43a}$$

$$x_2(t) = -\frac{1485}{722} + \frac{38115t^2}{27436} - \frac{1217865t^4}{4170272} + \frac{6573809t^6}{158470336} - \frac{57817309t^8}{12043745536} + \frac{70538244337t^{10}}{144163634065920} - \frac{20583954831t^{12}}{973905439023104} + \frac{41698599525t^{14}}{177250789902204928}, \tag{43b}$$

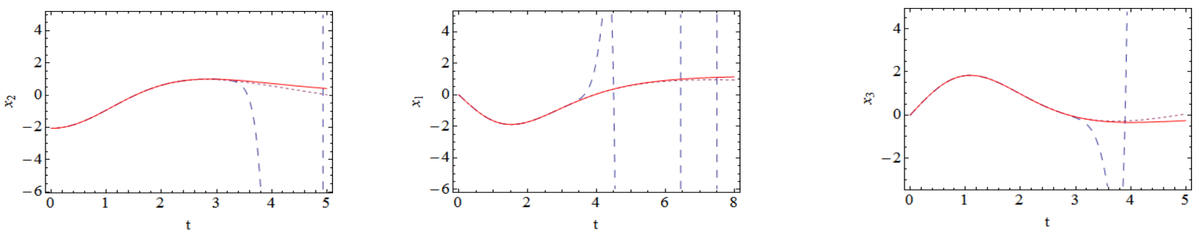
$$x_3(t) = \frac{38115t}{13718} - \frac{1217865t^3}{1042568} + \frac{19721427t^5}{79235168} - \frac{57817309t^7}{1505468192} + \frac{70538244337t^9}{14416363406592} - \frac{118900997457t^{11}}{243476359755776} + \frac{5981986395495t^{13}}{240554643438706688} - \frac{1880366689575t^{15}}{4570538225335427072}. \tag{43c}$$

The [5,5] order Padé approximation of the 10-order homotopy analytic approximate solution are

$$\frac{a_{1,i}\varphi_1(x) + a_{2,i}\varphi_2(x) + a_{3,i}\varphi_3(x) + a_{4,i}\varphi_4(x) + a_{5,i}\varphi_5(x)}{1 + b_{2,j}\varphi_2(x) + b_{3,j}\varphi_3(x) + b_{4,j}\varphi_4(x) + b_{5,j}\varphi_5(x)} \approx x_i(t).\,(i = 1,2,3). \tag{44}$$

We can get the coefficients of the equation (44) $\{a_{1,i}(t) \mid i = 1,\cdots,5\}, \{b_{1,j}(t) \mid j = 1,\cdots,5\}$.

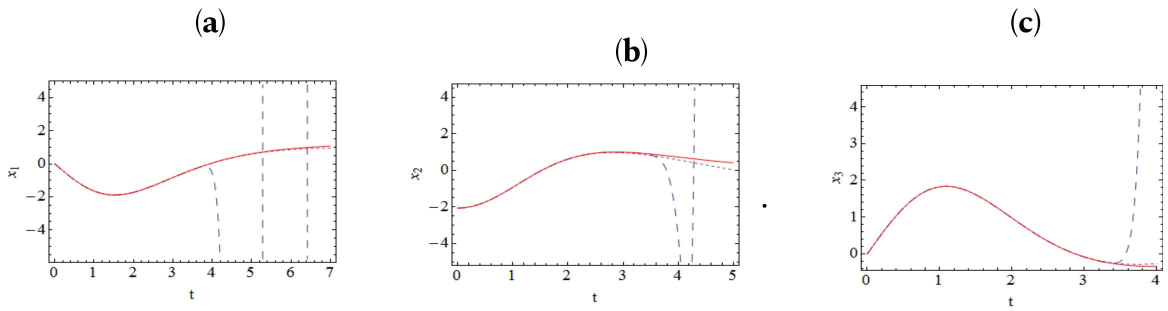
We provide 30 and 42 order the time history curve of [5,5] order Padé approximation compare with 30 and 42 order homotopy analytic approximate solution and exact solution. we can know that the [5,5] Padé approximate solutions provide effective agreement with the exact solutions in Figure 2 and Figure 3.



(**a**) (**b**) (**c**)

---30 order homotopy solution;___ exact solution;------Padé approximation solution

Figure 2. Comparison of 30th-order homotopy analysis approximation and exact solution for [5,5] order Padé approximation



42 order homotopy solution;____ exact solution;------Padé approximation solution

Figure 3. Comparison of 42th-order homotopy analysis approximation and exact solution for [5,5] order Padé approximation.

3.2 Example 2

In this part, we mainly consider the HAM for the homoclinic orbit of a externally and parametrically excited simply supported thin plate with planar dimensions a,b and thickness $\hbar$. The dimensionless equations of this plate system is indicated as

$$\ddot{x}_1 + \varepsilon\mu\dot{x}_1 + (\omega_1^2 + 2\varepsilon f_1 \cos\,\Omega_2 t)x_1 + \varepsilon(\alpha_1 x_1^3 + \alpha_2 x_1 x_2^2) = \varepsilon F_1 \cos\,\Omega_1 t, \tag{45a}$$

$$\ddot{x}_2 + \varepsilon\mu\dot{x}_2 + (\omega_2^2 + 2\varepsilon f_2 \cos\,\Omega_2 t)x_2 + \varepsilon(\beta_1 x_2^3 + \beta_2 x_1^2 x_2) = \varepsilon F_2 \cos\,\Omega_1 t, \tag{45b}$$

$$\alpha_1 = \frac{\lambda^2 + 81}{16\lambda^2}, \beta_1 = \frac{1}{16}\left(81\lambda^2 + \frac{1}{\lambda^2}\right), \quad \alpha_2 = \beta_2 = \frac{17\lambda^2}{(1+\lambda^2)^2} + \frac{625\lambda^2}{16(4+\lambda^2)^2} + \frac{625\lambda^2}{16(1+4\lambda^2)^2}. \tag{46}$$

The mathematical procedure and formulation of equation (46) and its coefficients $\varepsilon, \mu, \Omega_1, \Omega_2, f_1, f_2, F_1, F_2, \omega_1$ and ω_2 can be mentioned to Ref.[8] detailed. It is important to note that $\lambda = b/a$ in equation (46) is the aspect ratio of the thin plate.

According to the above system, we could consider the following nonlinear buckled thin plate system

$$\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = x_1 - x_1^3 - x_1 x_3^2, \frac{dx_3}{dt} = x_4, \frac{dx_4}{dt} = x_3 - x_3^3 - x_1^2 x_3, \tag{47}$$

in which, the initial condition of the above system as follows

$$x_1(0) = \frac{\sqrt{6}}{2}, x_2(0) = 0, x_3(0) = \frac{\sqrt{2}}{2}, x_4(0) = 0. \tag{48}$$

The exact homoclinic solution of equation (47) and (48) subject to the initial condition

$$\begin{aligned} x_1(t) &= \frac{\sqrt{6}}{2}\operatorname{sec\hbar} t, x_2(t) = -\frac{\sqrt{6}}{2}\operatorname{sec\hbar} t \tanh t \\ x_3(t) &= \frac{\sqrt{2}}{2}\operatorname{sec\hbar} t, x_4(t) = -\frac{\sqrt{2}}{2}\operatorname{sec\hbar} t \tanh t \end{aligned}. \tag{49}$$

We select orthonormal basis function $\{\varphi_i | i = 1,2,3,\dots\}$ as the base. According to the base of initial function, we choose initial guess as follows

$$\begin{aligned} x_{1,0}(0) &= \frac{\sqrt{6}}{2} - \frac{\sqrt{6}}{2}t^2 + \frac{\sqrt{6}}{2}t^3, x_{2,0}(0) = -\frac{\sqrt{6}}{2}t + \frac{3\sqrt{6}}{2}t^2 \\ x_{3,0}(0) &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t^2 + \frac{\sqrt{2}}{2}t^3, x_{4,0}(0) = \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}t^2 \end{aligned}. \tag{50}$$

The linear operators are defined as follows

$$L\begin{pmatrix} \varphi_1(t;q) \\ \varphi_2(t;q) \\ \varphi_3(t;q) \\ \varphi_4(t;q) \end{pmatrix} = \begin{pmatrix} \frac{\partial \varphi_1(t;q)}{\partial t} \\ \frac{\partial \varphi_2(t;q)}{\partial t} \\ \frac{\partial \varphi_3(t;q)}{\partial t} \\ \frac{\partial \varphi_4(t;q)}{\partial t} \end{pmatrix}. \tag{51}$$

The non-linear operators

$$N\begin{pmatrix} \varphi_1(t;q) \\ \varphi_2(t;q) \\ \varphi_3(t;q) \\ \varphi_4(t;q) \end{pmatrix} = \begin{pmatrix} \frac{\partial \varphi_1(t;q)}{\partial t} - \varphi_2(t;q) \\ \frac{\partial \varphi_2(t;q)}{\partial t} - \varphi_1(t;q) + [\varphi_1(t;q)]^3 + \varphi_1(t;q)\varphi_3(t;q)^2 \\ \frac{\partial \varphi_3(t;q)}{\partial t} - \varphi_4(t;q) \\ \frac{\partial \varphi_4(t;q)}{\partial t} - \varphi_3(t;q) + [\varphi_3(t;q)]^3 + \varphi_1^2(t;q)\varphi_3(t;q) \end{pmatrix} \tag{52}$$

Consequently, we can obtain the zero-order deformation equation of the system (47)

$$(1-q)L\begin{pmatrix}\varphi_1(t;q)-x_{1,0}(t)\\ \varphi_2(t;q)-x_{2,0}(t)\\ \varphi_3(t;q)-x_{3,0}(t)\\ \varphi_4(t;q)-x_{4,0}(t)\end{pmatrix} = q\hbar H(t)N\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\\ \varphi_3(t;q)\\ \varphi_4(t;q)\end{pmatrix}. \quad (53)$$

The m-order deformation equation

$$L\begin{pmatrix}x_{1,m}(t)-\chi_m x_{1,m-1}(t)\\ x_{2,m}(t)-\chi_m x_{2,m-1}(t)\\ x_{3,m}(t)-\chi_m x_{3,m-1}(t)\\ x_{4,m}(t)-\chi_m x_{4,m-1}(t)\end{pmatrix} = \hbar H(t)N\begin{pmatrix}R_{1,m}(x_{1,m-1})\\ R_{2,m}(x_{2,m-1})\\ R_{3,m}(x_{3,m-1})\\ R_{4,m}(x_{4,m-1})\end{pmatrix}. \quad (54)$$

Letting

$$H(t)=1, \quad (55)$$

From the basis of initial conditions and initial approximation, we can get

$$x_{1,m}(0)=0, x_{2,m}(0)=0, x_{3,m}(0)=0, x_{4,m}(0) = 0, (m\geq 1). \quad (56)$$

As well as

$$R_{1,m}(x_{1,m-1}) = x'_{1,m-1}(t)-x_{2,m-1}(t), \quad (57a)$$

$$R_{2,m}(x_{2,m-1}) = x'_{2,m-1}(t)-x_{1,m-1}(t)+\sum_{j+s=m-1}\left[x_{1,j}(t)\left(\sum_{i=0}^{s}x_{1,i}x_{1,s-i}\right)\right] + \sum_{j+s=m-1}\left[x_{1,j}(t)\left(\sum_{i=0}^{s}x_{3,i}x_{3,s-i}\right)\right], \quad (57b)$$

$$R_{3,m}(x_{3,m-1}) = x'_{3,m-1}(t)-x_{4,m-1}(t), \quad (57c)$$

$$R_{4,m}(x_{4,m-1}) = x'_{4,m-1}(t)-x_{3,m-1}(t)+\sum_{j+s=m-1}\left[x_{3,j}(t)\left(\sum_{i=0}^{s}x_{3,i}x_{3,s-i}\right)\right] + \sum_{j+s=m-1}\left[x_{3,j}(t)\left(\sum_{i=0}^{s}x_{1,i}x_{1,s-i}\right)\right]. \quad (57d)$$

By solving the m-order deformation equation, we have

$$x_{1,m}(t) = \chi_m x_{1,m-1}(t) + \hbar\int_0^t R_{1,m}(x_{1,m-1})ds, \quad (58a)$$

$$x_{2,m}(t) = \chi_m x_{2,m-1}(t) + \hbar \int_0^t R_{2,m}\big(x_{2,m-1}\big) ds, \tag{58b}$$

$$x_{3,m}(t) = \chi_m x_{3,m-1}(t) + \hbar \int_0^t R_{3,m}\big(x_{3,m-1}\big) ds, \tag{58c}$$

$$x_{4,m}(t) = \chi_m x_{4,m-1}(t) + \hbar \int_0^t R_{4,m}\big(x_{4,m-1}\big) ds. \tag{58d}$$

Therefore

$$x_{1,1}(t) = \sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2}} t^2 + \sqrt{\frac{3}{2}} t^3, \tag{59a}$$

$$x_{2,1}(t) = -\sqrt{6}t - \sqrt{\frac{3}{2}}\hbar t + 3\sqrt{\frac{3}{2}} t^2 + 3\sqrt{\frac{3}{2}}\hbar t^2 - \frac{5\hbar t^3}{\sqrt{6}} + \frac{5}{4}\sqrt{\frac{3}{2}}\hbar t^4 + \frac{3\sqrt{6}\hbar t^6}{5} -\sqrt{6}\hbar t^6 + \frac{2}{7}\sqrt{6}\hbar t^7 + \frac{3}{4}\sqrt{\frac{3}{2}}\hbar t^8 - \sqrt{\frac{2}{3}}\hbar t^9 + \frac{1}{5}\sqrt{\frac{3}{2}}\hbar t^{10}, \tag{59b}$$

$$x_{3,1}(t) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} t^2 + \frac{1}{\sqrt{2}} t^3, \tag{59c}$$

$$x_{4,1}(t) = -\sqrt{2}t - \frac{1}{\sqrt{2}}\hbar t + \frac{3t^2}{\sqrt{2}} + \frac{3\hbar t^2}{\sqrt{2}} - \frac{5\hbar t^3}{3\sqrt{2}} + \frac{5\hbar t^4}{4\sqrt{2}} + \frac{3}{5}\sqrt{2}\hbar t^5 -\sqrt{2}\hbar t^6 + \frac{2\sqrt{2}}{7}\hbar t^7 + \frac{3\hbar t^8}{4\sqrt{2}} - \frac{\sqrt{2}}{3}\hbar t^9 + \frac{\hbar t^{10}}{5\sqrt{2}}, \tag{59d}$$

So then, the m -order analytic approximation solution of the homoclinic orbit can be expressed as

$$x(t) = (x_1(t), x_2(t), x_3(t), x_4(t)), \tag{60}$$

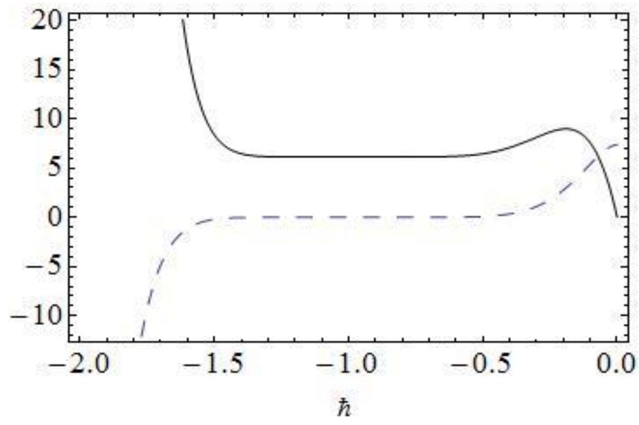
where

$$x_1(t) = x_{1,0}(t) + x_{1,1}(t) + x_{1,2}(t) + \cdots + x_{1,m}(t), \tag{61a}$$

$$x_2(t) = x_{2,0}(t) + x_{2,1}(t) + x_{2,2}(t) + \cdots + x_{2,m}(t), \tag{61b}$$

$$x_3(t) = x_{3,0}(t) + x_{3,1}(t) + x_{3,2}(t) + \cdots + x_{3,m}(t), \tag{61c}$$

$$x_4(t) = x_{4,0}(t) + x_{4,1}(t) + x_{4,2}(t) + \cdots + x_{4,m}(t). \tag{61d}$$



------- $x_1^m(0) \sim \hbar$; ____ $x_2^m(0) \sim \hbar$.

Figure 4. $\hbar$ -curves of $x_1^m(0)$ and $x_2^m(0)$ obtained from 10th-order approximation for Eq. (47).

In this section, the $\hbar$ -curve is drawn in Figure4 to make sure the efficacious range of $\hbar$. In the valid range of $\hbar$, we can ensure the convergence of the analytic solution. It is clear that the series of $x_1^m(0)$ and $x_2^m(0)$ is convergent in the range of $-1.5 < \hbar < -0.2$. Where, we choose $h = -1$. So, we can get the 10-order analytic approximate solution in the way of HAM, which are

$$
\begin{aligned}
x_1(t) = {} & \sqrt{\frac{3}{2}} - \frac{1}{2}\sqrt{\frac{3}{2}}t^2 + \frac{5t^4}{8\sqrt{6}} - \frac{61t^6}{240\sqrt{6}} + \frac{61t^8}{1344\sqrt{6}} + \frac{515t^9}{4032\sqrt{6}} - \frac{19t^{10}}{1400\sqrt{6}} - \frac{233\sqrt{3}t^{11}}{2200\sqrt{2}} \\
& + \frac{3631t^{12}}{9240\sqrt{6}} - \frac{4097t^{13}}{34320\sqrt{6}} - \frac{719863t^{14}}{4204200\sqrt{2}} + \frac{13927t^{15}}{63700\sqrt{6}} - \frac{1278049t^{16}}{14014000\sqrt{6}} \\
& - \frac{454219\sqrt{3}t^{17}}{38118080\sqrt{2}} + \frac{2025979t^{18}}{28588560\sqrt{6}} - \frac{140350949t^{19}}{3621217600\sqrt{6}} + \frac{2223713t^{20}}{1597596000\sqrt{6}} \\
& + \frac{61579207t^{21}}{5761028000\sqrt{6}} - \frac{33465931t^{22}}{5503297800\sqrt{6}} + \frac{6189259\sqrt{3}t^{23}}{35237232800\sqrt{2}} + \frac{3342023\sqrt{3}t^{24}}{10067780800\sqrt{2}} \\
& - \frac{5023\sqrt{3}t^{25}}{25840000\sqrt{2}} + \frac{56\sqrt{2}t^{26}}{747175\sqrt{3}} - \frac{56\sqrt{2}t^{27}}{6724575\sqrt{3}}
\end{aligned},
$$

$$
\begin{aligned}
x_2(t) = {} & -\sqrt{\frac{3}{2}}t + \frac{5t^3}{2\sqrt{6}} - \frac{61t^5}{40\sqrt{6}} + \frac{61t^7}{168\sqrt{6}} + \frac{515t^8}{488\sqrt{6}} - \frac{19t^9}{140\sqrt{6}} - \frac{233\sqrt{3}t^{10}}{200\sqrt{2}} + \frac{3631t^{11}}{770\sqrt{6}} \\
& - \frac{4097t^{12}}{2640\sqrt{6}} - \frac{719863t^{13}}{300300\sqrt{6}} + \frac{13927\sqrt{3}t^{14}}{12740\sqrt{2}} - \frac{1278049t^{15}}{875875\sqrt{6}} - \frac{454219\sqrt{3}t^{16}}{2242240\sqrt{2}} \\
& + \frac{2025979\sqrt{3}t^{17}}{4764760\sqrt{2}} - \frac{140350949t^{18}}{190590400\sqrt{6}} + \frac{2223713t^{19}}{79879800\sqrt{6}} + \frac{61579207\sqrt{3}t^{20}}{823004000\sqrt{2}} - \frac{33465931t^{21}}{250149900\sqrt{6}} \\
& + \frac{6189256\sqrt{3}t^{22}}{1532053600\sqrt{2}} + \frac{10026069\sqrt{3}t^{23}}{1258472600\sqrt{2}} - \frac{5023\sqrt{3}t^{24}}{1033600\sqrt{6}} + \frac{112\sqrt{2}t^{25}}{57475\sqrt{3}} - \frac{56\sqrt{6}t^{26}}{747175}
\end{aligned},
$$

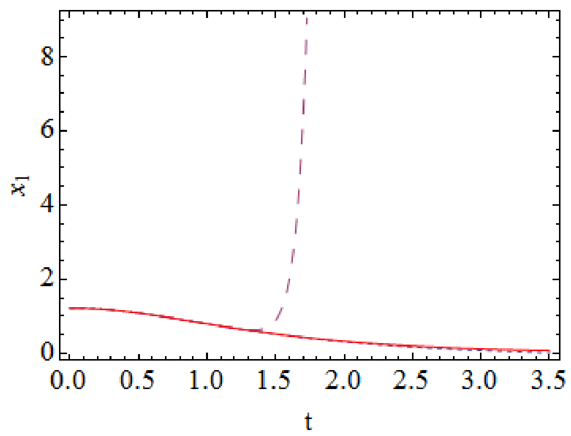
$$x_3(t) = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}t^2 + \frac{5t^4}{24\sqrt{2}} - \frac{61t^6}{720\sqrt{2}} + \frac{61t^8}{4032\sqrt{2}} + \frac{515t^9}{12096\sqrt{2}} - \frac{19t^{10}}{4200\sqrt{2}} - \frac{233t^{11}}{2200\sqrt{2}} + \frac{3631t^{12}}{27720\sqrt{2}} - \frac{4097t^{13}}{102960\sqrt{2}} - \frac{719863t^{14}}{12612600\sqrt{2}} + \frac{13927t^{15}}{191100\sqrt{2}} - \frac{1278049t^{16}}{42042000\sqrt{2}} - \frac{454219t^{17}}{38118080\sqrt{2}} + \frac{2025979t^{18}}{85765680\sqrt{2}} - \frac{140350949t^{19}}{10863652800\sqrt{2}} + \frac{2223713t^{20}}{4792788000\sqrt{2}} + \frac{61579207t^{21}}{17283084000\sqrt{2}} - \frac{33465931t^{22}}{16509893400\sqrt{2}} + \frac{6189259t^{23}}{35237232800\sqrt{2}} + \frac{3342023t^{24}}{10067780800\sqrt{2}} - \frac{5023t^{25}}{25840000\sqrt{2}} + \frac{56\sqrt{2}t^{26}}{2241525} - \frac{56\sqrt{2}t^{27}}{20173725},$$

$$x_4(t) = -\frac{1}{\sqrt{2}}t + \frac{5t^3}{6\sqrt{2}} - \frac{61t^5}{120\sqrt{2}} + \frac{61t^7}{504\sqrt{2}} + \frac{515t^8}{1344\sqrt{2}} - \frac{19t^9}{420\sqrt{2}} - \frac{233t^{10}}{200\sqrt{2}} + \frac{3631t^{11}}{2310\sqrt{2}} - \frac{719863t^{13}}{900900\sqrt{2}} + \frac{13927t^{14}}{12740\sqrt{2}} - \frac{1278049t^{15}}{2627625\sqrt{2}} - \frac{454219t^{16}}{2242240\sqrt{2}} + \frac{2025979t^{17}}{4764760\sqrt{2}} - \frac{140350949t^{18}}{571771200\sqrt{2}} + \frac{2223713t^{19}}{239639400\sqrt{2}} + \frac{61579207t^{20}}{823004000\sqrt{2}} - \frac{33465931t^{21}}{750449700\sqrt{2}} + \frac{6189259t^{22}}{1532053600\sqrt{2}} + \frac{10026069t^{23}}{1258472600\sqrt{2}} - \frac{5023t^{24}}{1033600\sqrt{2}} + \frac{112\sqrt{2}t^{25}}{172425} - \frac{56\sqrt{2}t^{25}}{747175}$$

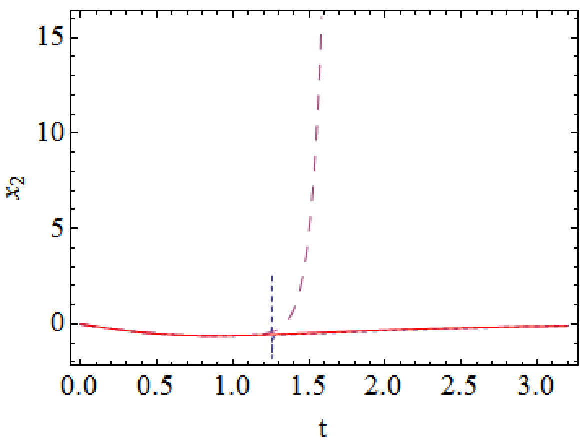
The [4,4] order Padé approximation of the 8-order homotopy analytic approximate solution are

$$\frac{a_{1,i}\varphi_1(x) + a_{2,i}\varphi_2(x) + a_{3,i}\varphi_3(x) + a_{4,i}\varphi_4(x)}{1 + b_{2,j}\varphi_2(x) + b_{3,j}\varphi_3(x) + b_{4,j}\varphi_4(x)} \approx x_i(t), \quad (63)$$

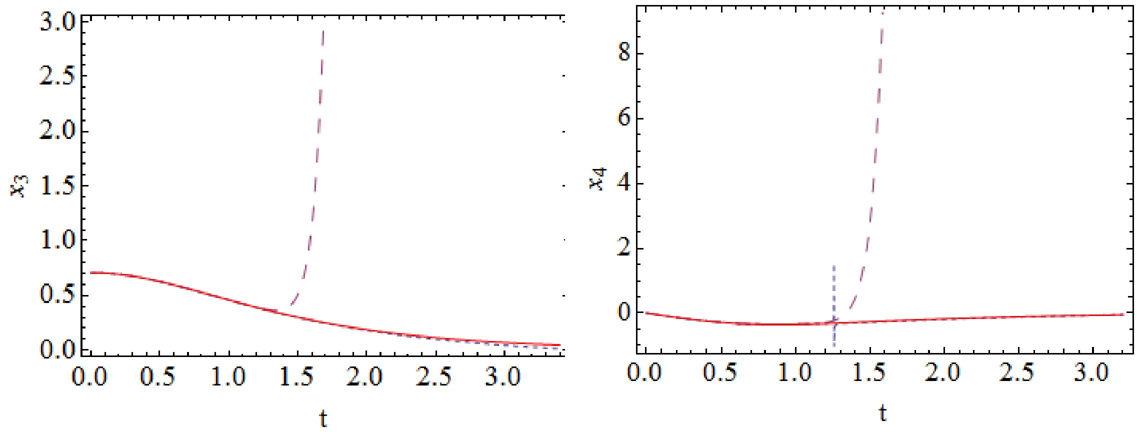
in the same way, we can get the coefficients of the equation (63). Where, $\{\varphi_i(t)|i = 1,2,\ldots\}$ is orthonormal basis function. By comparison, we could find that Padé approximation of homotopy solution provide excellent agreement with the exact solution in Figure 5.



(a)



(b)



(c) (d)

- - - - 8 order homotopy solution; ____ exact solution; --------[4,4]Padé approximation solution

Figure 5. Comparison of 8th-order homotopy analysis approximation and exact solution for [4,4] order Padé approximation.

4. Conclusion

In this paper, the analytic approximate solution of non-linear system is obtained by HAM. Besides, we provide the process of solving the non-linear system in the HAM. In addition, we offer an ingenious method to control the convergence of the approximation solution. For another, we provide Padé approximation of the homotopy analytic approximate solution. By comparison, it is proved that Padé approximation solution is more excellent than homotopy analytic approximate.

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REFERENCES

[1] Li, Y.; McLaughlin, D. W. (1997) Homoclinic orbits and chaos in discretized perturbed NLS systems: Part I. Homoclinic orbits. *Journal of Nonlinear Science*, *7*, 211-269.

[2] Roberto, C.; Gregor, K.; Tin, S.K. (1998) A Melnikov method for homoclinic orbits with many pulses. *Archive for Rational Mechanics and Analysis*, *143*, 105-193.

[3] Shi, Y. X. (2017) Melnikov analysis of chaos and heteroclinic bifurcation in Josephson system driven by an amplitude-modulated force. *International Journal of Dynamics and Control*, *6*, 1–12.

[4] Wang, M. N.; Xu, Z. Y. (2001) On the homoclinic orbits in a class of two-degree-of-freedom systems under the resonance conditions. *Applied Mathematicas and Mechanics*, *22*, 340-352.

[5] Beyn, W. J.; Hüls, T. (2004) Error estimates for approximating non-hyperbolic heteroclinic orbits of maps. *Numerische Mathematik*, *99*, 289-323.

[6] Koltsova, O. Yu. (2005) On hamiltonian systems with a homoclinic orbit to a saddle-center. *Journal of Mathematical Sciences*, *128*, 2787-2790.

[7] Wilczak, D. (2006) The existence of Shilnikov homoclinic orbits in the Michelson system: a computer assisted proof. *Foundations of Computational Mathematicas*, *6*, 495-535.

[8] Deng, X. Q.; Cheng, G.; Shi, H. P. (2009) Subharmonic solutions and homoclinic orbits of second order discrete hamiltonian systems with potential changing sign. *Acta Applicandae Mathematicae*, *58*, 1198-1206.

[9] Chen, Y. Y.; Chen, S. H. (2009) Homoclinic and heteroclinic solutions of cubic strongly nonlinear autonomous oscillators by the hyperbolic perturbation method. *Nonlinear Dynamics*, *58*, 417.

[10] Chen, S. H.; Chen, Y. Y.; Sze, K. Y. (2010) Homoclinic and heteroclinic solutions of cubic strongly nonlinear autonomous oscillators by hyperbolic Lindstedt-Poincaré method. *Science China Technological Sciences*, *53*, 692–702.

[11] Deng, X.; Liu, X.; Shi, H.; Zhou, T. (2011) Homoclinic orbits for second order nonlinear p-Laplacian difference equations. *Journal of Contemporary Mathematical Analysis*, *46*, 172-182.

[12] Shi, X. B.; Zheng, Q. F.; Zhang, Q. M. (2013) Existence of homoclinic orbits for a class of p-Laplacian systems in a weighted Sobolev space. *Boundary Value Problems*, *2013*, 1-18.

[13] Huang, W. N.; Tang, X. H. (2015) The existence of heteroclinic orbits for a class of the second-order hamiltonian system. *Mediterranean Journal of Mathematics*, *12*, 9–20.

[14] Li, Z.; Tang, J.; Cai, P. (2016) Predicting homoclinic and heteroclinic bifurcation of generalized Duffing-Harmonic-Van Der Pol oscillator. *Qualitative Theory of Dynamical Systems*, *15*, 1-19.

[15] Liao, S. J. (2004) On the homotopy analysis method for nonlinear problems. *Applied Mathematics and Computation*, *147*, 499-513.

[16] Ganjiani, M.; Ganjiani, H. (2009) Solution of coupled system of nonlinear differential equations using homotopy analysis method. *Nonlinear Dynamics*, *56*, 159-167.

[17] Zhang, W.; Qian, Y. H.; Lai, S. K. (2012) Extended homotopy analysis method for multi-degree-of-freedom non-autonomous nonlinear dynamical systems and its application. *Acta Mechanica*, *223*, 2537-2548.

[18] Qian, L. H.; Qian, Y. H.; Chen, S. H. (2014) Homotopy analysis method for homoclinic orbit of a buckled thin plate system. *Acta Mechanica*, *225*, 373-381.

[19] Liu, W. K.; Qian, Y. H. (2014) Homotopy analysis method for heteroclinic orbits of Michelson system. *Dynamical Systems and Control*, *3*, 29-37.

[20] Tajaddodianfar, F.; Yazdi, M. R. H.; Pishkenari, H. N. (2017) Nonlinear dynamics of MEMS/NEMS resonators: analytical solution by the homotopy analysis method. *Microsystem Technologies*, *23*, 1913-1926.

[21] Ismail, M. H.; Matalgah, M. M. (2006) On the use of Padé approximation for performance evaluation of maximal ratio combining diversity over weibull fading channel. *EURASIP Journal On Wireless Communications and Networking*, *2006*, 1-7.

[22] Kandayan, A. A. (2009) Multipoint Padé approximations of the beta function. *Mathematical Notes*, *85*, 176–189.

[23] Kandayan, A. A.; Sorokin, V. N. (2010) Multipoint Hermite—Padé approximations for beta functions. *Mathematical Notes*, *87*, 204–217.

[24] Starovoitov, A. P. (2014) The asymptotic form of the Hermite-Padé approximations for a system of mittag-leffler functions. *Russian Mathematics*, *58*, 49–56.

[25] Qian, Y. H.; Fu, H. X.; Shen, L. (2016) Padé approximation and its Application in solving nonlinear systems. *Dynamical Systems and Control*, *5*, 161-178.