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Intrinsic Square Functions and Commutators on Herz Spaces With Variable Exponents

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ABSTRACT. In this article, we will study the boundedness of intrinsic square functions on the Herz spaces $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. The corresponding commutators generated by $BMO(\mathbb{R}^n)$ functions and intrinsic square functions are also discussed on the aforementioned Herz spaces.

1. INTRODUCTION

For $0 < \gamma \leq 1$, let C_γ be the family of all functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that φ has support contained in $\{x \in \mathbb{R}^n : |x| \leq 1\}$, $\int_{\mathbb{R}^n} \varphi(x)dx = 0$ and such that for any $x_1, x_2 \in \mathbb{R}$ the following inequality holds:

$$|\varphi(x_1) - \varphi(x_2)| \leq |x_1 - x_2|^\gamma. \tag{1}$$

For $(y,t) \in \mathbb{R}^{n+1}_+$ and $f \in L_{\text{loc}}(\mathbb{R}^n)$, set

$$A_\gamma(f)(y,t) = \sup_{\varphi \in C_\gamma} |f * \varphi_t(y)| = \sup_{\varphi \in C_\gamma} \left| \int_{\mathbb{R}^n} \varphi_t(y-z) f(z) \mathrm{d}z \right|. \tag{2}$$

The intrinsic square function of f of order γ is defined by

$$S_\gamma(f)(x) = \left(\iint_{\Gamma(x)} (A_\gamma(f)(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}, \tag{3}$$

where $\varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t})$ and $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$.

The definition of intrinsic square function S_γ was first introduced by Wilson in [1,2]. In [2], Wilson proved the weighed L^p- boundedness of intrinsic square functions. Lerner [3] proved sharp $L^p(w)$ norm inequalities for the intrinsic square function in terms of the A_p characteristic of w for all $1 < p < \infty$. The boundedness of intrinsic Littlewood-Paley functions on Musielak-Orlicz Morrey and Campanato spaces was considered in [4].

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Let $b \in L_{\text{loc}}(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$. The commutator generated by b and the intrinsic square function $S_\gamma(f)(x)$ is defined by

$$[b, S_\gamma](f)(x) = \left(\int\int_{\Gamma(x)} \sup_{\varphi \in C_\gamma} \left|\int_{\mathbb{R}^n} (b(x) - b(z))\varphi_t(y-z)f(z)\mathrm{d}z\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}. \quad (4)$$

Wang [5], established the commutators of intrinsic square functions $[b, S_\gamma]$ on weighted L^p space. Guliyev et al. [6] proved the intrinsic square function and their commutator on weighted Orlicz-Morrey space.

Moreover, Izuki [7] defined the Herz spaces with one variable exponent $p(\cdot)$ and investigated the boundedness of sublinear operators on those space. Wang and Tao [8] generalized the Herz spaces $\dot{K}^{\alpha,p(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ with two variable exponent $q(\cdot)$ and $p(\cdot)$ and obtained some boundedness results for Littlewood-Paley operators and their commutators in these spaces. The boundedness of fractional integral with variable kernel on Herz spaces with variable exponents was considered in [9]. In [10] the author proved the boundedness of the commutator of the intrinsic square function in variable exponent spaces.

The aim of this paper is to discuss boundedness properties of intrinsic square functions and their commutators on the Herz spaces $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$.

2. Mathematical background

Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with measure $|E| > 0$. Let us denote by χ_E the characteristic function of E. We mention that, throughout the paper, C denotes a positive constant, not necessarily the same at each occurrence.

We recall some definitions.

Definition 2.1 ([11], Chapter 2, p. 18). Let $p(\cdot) : E \to [1, \infty)$ be a measurable function. The *variable exponent Lebesgue space* is defined by

$$L^{p(\cdot)}(E) = \left\{ f \text{ is measurable} : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \eta > 0 \right\}.$$

The space $L^{p(\cdot)}_{\text{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(E) = \{f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for any compact set } K \subset E\}.$$

The Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach spaces with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\eta > 0 : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x \le 1\right\}.$$

We denote $p_- = \text{ess inf}\{p(x) : x \in E\}$, $p_+ = \text{ ess sup}\{p(x) : x \in E\}$. $\mathcal{P}(E)$ is the set of all measurable functions $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < +\infty$ and $\mathcal{P}^0(E)$ is the set of all measurable functions $p(\cdot)$ satisfying $p_- > 0$ and $p_+ < +\infty$

If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies the follows inequalities:

$$|p(x) - p(y)| \leq \frac{-C}{\log(|x-y|)}, |x-y| \leq 1/2;$$
$$|p(x) - p(y)| \leq \frac{C}{\log(e+|x|)}, |y| \geq |x|.$$

then, we have $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$.

We know that, if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then the Hardy−Littlewood maximal operator M,

$$Mf(x) = \sup_{B \subseteq \mathbb{R}^n} \frac{1}{|B|} \int_B |f(y)| dy,$$

being B a sphere in $\mathbb{R}^n$, is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. The set $\mathcal{B}(\mathbb{R}^n)$ consists of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Let us now recall the definition of space $BMO(\mathbb{R}^n)$. This space consists of all locally integrable functions f such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \|f\|_* = \sup_Q |Q|^{-1} \int_Q |f(x) - f_Q| dx < \infty,$$

where $f_Q = |Q|^{-1} \int_Q f(y)dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of Q.

Now, we give the definition of Herz space with variable exponents $q(\cdot), p(\cdot)$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}, C_k = B_k \backslash B_{k-1}, \chi_k = \chi_{C_k}, k \in \mathbb{Z}$.

Definition 2.2 ([8]). Let $\alpha \in \mathbb{R}^n$, $q(\cdot), p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponents $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \left\|\{2^{k\alpha}|f\chi_k|\}_{k=-\infty}^{\infty}\right\|_{l^{q(\cdot)}(L^{p(\cdot)})}$$
$$= \inf\left\{\beta > 0 : \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|f\chi_k|}{\beta} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1 \right\}.$$

The nonhomogeneous Herz space with variable exponents $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{K^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \left\|\{2^{k\alpha}|f\chi_k|\}_{k=0}^{\infty}\right\|_{l^{q(\cdot)}(L^{p(\cdot)})}$$
$$= \inf\left\{\beta > 0 : \sum_{k=0}^{\infty} \left\| \left(\frac{2^{k\alpha}|f\chi_k|}{\beta} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1 \right\}.$$

Next, we need some Lammas that will be used in the proofs of our main results.

Lemma 2.3 ([11]). *(Generalized Hölder's inequality) If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then there exists a constant C such that, for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, the following inequality holds:*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $C = 1 + \frac{1}{p_-} - \frac{1}{p_+}$.

Lemma 2.4 ([12]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then, there exists a constant $C > 0$ such that for any balls $B \subset \mathbb{R}^n$, the following inequality holds:*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.5 ([12]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. For $h = 1, 2$, there exist constants $\delta_{h1}, \delta_{h2}, C > 0$ such that for all balls $B \subset \mathbb{R}^n$ and all measurable $S \subset B$ the following inequalities hold:*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|}, \ \frac{\|\chi_S\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_{h1}}, \ \frac{\|\chi_S\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_{h2}}.$$

Lemma 2.6 ([8]). *Let $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ and $f \in L^{p(\cdot)q(\cdot)}(\mathbb{R}^n)$. Then*

$$\min(\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \ \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}) \leq \||f|^{q(\cdot)}\|_{L^{p(\cdot)}} \leq \max(\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \ \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}).$$

Lemma 2.7 ([12]). *Let us assume that $b \in BMO(\mathbb{R}^n)$ and that n is a positive integer. Then there exists a constant $C > 0$, such that for any $k, j \in \mathbb{Z}$ with $k > j$, the following inequalities hold:*

(1) $C^{-1}\|b\|_* \leq \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b - b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*$,

(2) $\|(b - b_{B_j})\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(k - j)\|b\|_*\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

3. Boundedness of the intrinsic square functions

Let $1 < p < \infty$, $p' = \frac{p}{p-1}$ and let w be a weight (i.e., a nonnegative locally integrable function on $\mathbb{R}^n$). We say that $w \in A_p$ if there exists $C > 0$ such that for every cube $Q \subseteq \mathbb{R}^n$, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\mathrm{d}x\right)^{p-1} \leq C < \infty.$$

Wilson [1] proved the following weighted ($L^p - L^p$) boundedness of the intrinsic square functions.

Lemma 3.1 ([1]). *Let $1 < p < \infty$, $0 < \gamma \leq 1$ and $w \in A_p$. Then there exists a constant $C > 0$ such that*

$$\|S_\gamma(f)\|_{L^p_w} \leq C\|f\|_{L^p_w}.$$

Lemma 3.2 ([13]). *Given a family of functions $\mathcal{F}$, assume that for p_0, $1 < p_0 < \infty$ and every $w_0 \in A_{p_0}$,*

$$\int_{\mathbb{R}^n} f(x)^{p_0}w_0(x)\mathrm{d}x \leq \int_{\mathbb{R}^n} g(x)^{p_0}w_0(x)\mathrm{d}x, \qquad (f, g) \in \mathcal{F}.$$

If $p(\cdot) \in \mathcal{P}(E)$, then for all $(f, g) \in \mathcal{F}$ and $f \in L^{p(\cdot)}(E)$ we have

$$\|f\|_{L^{p(\cdot)}(E)} \leq C\|g\|_{L^{p(\cdot)}(E)}.$$

Since $A_{p/s'} \subset A_\infty$, by applying Lemma 3.1 and Lemma 3.2, it is easy to get the boundedness of the intrinsic square functions S_γ on $L^{p(\cdot)}$.

Theorem 3.3. *Let us assume that $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \geq (q_1)_+$ and $0 < \gamma \leq 1$. If $-n\delta_{12} < \alpha < n\delta_{11}$, where δ_{11}, δ_{12} are the constants in Lemma 2.5, then the operator S_γ is bounded from $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n\right)$ to $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n\right)$.*

Before starting the proof of Theorem 3.3, we state a simple inequality that will be used in the proof.

Remark. Let $h \in \mathbb{N}$, $a_h \geq 0$, $1 \leq p_h < \infty$. We have

$$\sum_{h=0}^{\infty} a_h^{p_h} \leq \left(\sum_{h=0}^{\infty} a_h\right)^{p_*},$$

here

$$p_* = \begin{cases} \min\limits_{h\in\mathbb{N}} p_h & \text{if } \sum\limits_{h=0}^{\infty} a_h \leq 1, \\ \max\limits_{h\in\mathbb{N}} p_h & \text{if } \sum\limits_{h=0}^{\infty} a_h > 1. \end{cases}$$

Proof. We give the proof in the homogeneous case $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. The same proof is also valid for the nonhomogeneous case $K^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$.

Let $f \in \dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$. We decompose f as follows:

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

By the definition of the norm in $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have

$$\|S_\gamma(f)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|S_\gamma(f)\chi_k|}{\beta}\right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq 1\right\}.$$

We have

$$\left\| \left(\frac{2^{k\alpha}|S_\gamma(f)\chi_k|}{\beta}\right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq \left\| \left(\frac{2^{k\alpha}|\sum\limits_{j=-\infty}^{\infty} S_\gamma(f_j)\chi_k|}{\beta_{11}+\beta_{12}+\beta_{13}}\right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\leq C \left\| \left(\frac{2^{k\alpha}|\sum\limits_{j=-\infty}^{k-2} S_\gamma(f_j)\chi_k|}{\beta_{11}}\right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} + C \left\| \left(\frac{2^{k\alpha}|\sum\limits_{j=k-1}^{k+1} S_\gamma(f_j)\chi_k|}{\beta_{12}}\right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$+C \left\| \left(\frac{2^{k\alpha}|\sum\limits_{j=k+2}^{\infty} S_\gamma(f_j)\chi_k|}{\beta_{13}}\right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}},$$

where

$$\beta_{11} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=-\infty}^{k-2} S_\gamma(f_j)\chi_k \right| \right\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

$$\beta_{12} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=k-1}^{k+1} S_\gamma(f_j)\chi_k \right| \right\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

$$\beta_{13} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=k+2}^{\infty} S_\gamma(f_j)\chi_k \right| \right\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})}.$$

If $\beta_1 = \beta_{11} + \beta_{12} + \beta_{13}$ thus

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|S_\gamma(f_j)\chi_k|}{\beta_1} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C.$$

Then

$$\|S_\gamma(f)\chi_k\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} \leq C\beta_1 \leq C[\beta_{11} + \beta_{12} + \beta_{13}].$$

Hence, if we prove that

$$\beta_{11} \leq C\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}, \qquad \beta_{12} \leq C\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}, \qquad \beta_{13} \leq C\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)},$$

we are done. Let us set $\beta_1 = \|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$.

We consider β_{12} first. Applying Lemma 2.6, we have

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=k-1}^{k+1} S_\gamma(f_j)\chi_k \right|}{\beta_1} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\ &\leq \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} \left| \sum_{j=k-1}^{k+1} S_\gamma(f_j)\chi_k \right|}{\beta_1} \right\|_{L^{p_1(\cdot)}}^{(q_2^1)_k} \\ &\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{k+1} \left\| \frac{2^{k\alpha}|S_\gamma(f_j)\chi_k|}{\beta_1} \right\|_{L^{p_1(\cdot)}} \right)^{(q_2^1)_k}, \end{aligned}$$

where

$$(q_2^1)_k = \begin{cases} (q_2)_- & \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=k-1}^{k+1} S_\gamma(f_j)\chi_k \right|}{\beta_1} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq 1, \\ (q_2)_+ & \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=k-1}^{k+1} S_\gamma(f_j)\chi_k \right|}{\beta_1} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

By the boundedness of S_γ on $L^{p(\cdot)}(\mathbb{R}^n)$, it follows

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-1}^{k+1}S_\gamma(f_j)\chi_k|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C\sum_{k=-\infty}^{\infty}\sum_{j=k-1}^{k+1}\left\|\frac{2^{k\alpha}|f_j|}{\beta_1}\right\|_{L^{p_1(\cdot)}}^{(q_2^1)_k}.$$

Since $f\in \dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$, then we get $\left\|\frac{2^{j\alpha}|f\chi_j|}{\beta_1}\right\|_{L^{p_1(\cdot)}}\leq 1$, and $\sum\limits_{j=-\infty}^{\infty}\left\|\left(\frac{2^{j\alpha}|f\chi_j|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\leq 1$. Hence, again applying Lemma 2.6, we obtain that

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-1}^{k+1}S_\gamma(f_j)\chi_k|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} &\leq C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2^1)_k}{(q_1)_+}}\\ &\leq C\left[\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right]^{q_*}\leq C.\end{aligned}$$

Here $q_*=\min\limits_{k\in\mathbb{N}}\frac{(q_2^1)_k}{(q_1)_+}$.
The previous calculations imply that

$$\beta_{12}\leq C\beta_1\leq C\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Next we estimate β_{11}. If $x\in C_k$, $(y,t)\in\Gamma(x)$, $z\in C_j\cap\{z:|y-z|\leq t\}$, $j\leq k-2$, then

$$t\geq\frac{1}{2}(|x-y|+|y-z|)\geq\frac{1}{2}|x-z|\geq\frac{1}{4}|x|.$$

Thus, we have

$$\begin{aligned}|A_\gamma(f_j)(x)| &= \left(\int\int_{\Gamma(x)}\sup_{\varphi\in C_\gamma}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)f_j(z)\mathrm{d}z\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\ &\leq C\left(\int_{\frac{|x|}{4}}^{\infty}\int_{|x-y|<t}\left|t^{-n}\int_{C_j\bigcap\{z:|y-z|\leq t\}}\varphi_t(y-z)f_j(z)\mathrm{d}z\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\ &\leq C\left(\int_{C_j}|f_j(z)|dz\right)\left(\int_{\frac{|x|}{4}}^{\infty}\frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}}\\ &\leq C2^{-kn}\int_{C_j}|f_j(z)|dz\\ &= C2^{-kn}\|f_j\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Then, by using Lemmas 2.4 - 2.6 and $\left\|\frac{2^{j\alpha}|f\chi_j|}{\beta_1}\right\|_{L^{p_1(\cdot)q_1(\cdot)}} \leq 1$, we deduce that

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum\limits_{j=-\infty}^{k-2}S_\gamma(f_j)\chi_k|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\
&\leq C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}|\sum\limits_{j=-\infty}^{k-2}S_\gamma(f_j)\chi_k|}{\beta_1}\right\|_{L^{p_1(\cdot)}}^{(q_2^2)_k}\\
&\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{-kn}\left\|\frac{f_j}{\beta_1}\right\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{p_1(\cdot)}}\right]^{(q_2^2)_k}\\
&\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}\left\|\frac{f\chi_j}{\beta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}2^{-kn}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right]^{(q_2^2)_k}\\
&\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}\left\|\frac{f\chi_j}{\beta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}\right]^{(q_2^2)_k}\\
&\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_{11}}\left\|\frac{f\chi_j}{\beta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^2)_k}\\
&\leq C\sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_{11}-\alpha)}\left\|\left(\frac{|2^{\alpha j}f\chi_j|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}}\right]^{(q_2^2)_k},
\end{aligned}
$$

where

$$
(q_2^2)_k=\begin{cases}(q_2)_- & \left\|\left(\frac{2^{k\alpha}|\sum\limits_{j=-\infty}^{k-2}S_\gamma(f_j)\chi_k|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1,\\ (q_2)_+ & \left\|\left(\frac{2^{k\alpha}|\sum\limits_{j=-\infty}^{k-2}S_\gamma(f_j)\chi_k|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}> 1.\end{cases}
$$

If $(q_1)_+ < 1$, then by the fact $(p_1)_+ \leq (p_2)_- \leq (q_2^1)_k$, we have

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum\limits_{j=-\infty}^{k-2}S_\gamma(f_j)\chi_k|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\
&\leq C\sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_{11}-\alpha)}\left\|\left(\frac{|2^{\alpha j}f\chi_j|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{\frac{(q_2^2)_k}{(q_1)_+}}
\end{aligned}
$$

$$\leq C\left[\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{\alpha j}f\chi_j|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_{11}-\alpha)}\right]^{q_*}\leq C,$$

here $q_*=\min_{k\in\mathbb{N}}\frac{(q_2^2)_k}{(q_1)_+}$.

If $(q_1)_+\geq 1$, applying Hölder's inequality, it follows that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}S_\gamma(f_j)\chi_k|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_{11}-\alpha)\frac{(q_2)_+}{2}}\left\|\left(\frac{|2^{\alpha j}f\chi_j|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{\frac{(q_2^2)_k}{(q_1)_+}}$$

$$\times\left[\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_{11}-\alpha)\frac{((q_1)_+)'}{2}}\right]^{\frac{(q_2^2)_+}{((q_1)_+)'}}$$

$$\leq C\left[\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{\alpha j}f\chi_j|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_{11}-\alpha)\frac{(q_1)_+}{2}}\right]^{q_*}\leq C,$$

here $q_*=\min_{k\in\mathbb{N}}\frac{(q_2^2)_k}{(q_1)_+}$.

Then, from the above calculations it follows that

$$\beta_{11}\leq C\beta_1\leq C\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Finally, we estimate β_{13}. If $x\in C_k$, $(y,t)\in\Gamma(x)$, $z\in C_j\cap\{z:|y-z|\leq t\}$, $j\geq k+2$, then

$$t\geq\frac{1}{2}(|x-y|+|y-z|)\geq\frac{1}{2}|x-z|\geq\frac{1}{2}(|z|-|x|)\geq\frac{1}{4}|z|.$$

Thus, we have

$$\begin{aligned}|A_\gamma(f_j)(x)|&=\left(\int\int_{\Gamma(x)}\sup_{\varphi\in C_\gamma}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)f_j(z)\mathrm{d}z\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\&\leq C\left(\int_{\frac{|x|}{4}}^{\infty}\int_{|x-y|<t}\left|t^{-n}\int_{C_j\bigcap\{z:|y-z|\leq t\}}\varphi_t(y-z)f_j(z)\mathrm{d}z\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\&\leq C\left(\int_{C_j}|f_j(z)|dz\right)\left(\int_{\frac{|x|}{4}}^{\infty}\frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}}\\&\leq C2^{-jn}\int_{C_j}|f_j(z)|dz\\&=C2^{-jn}\|f_j\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Thus, from Lemmas 2.4 - 2.6 and $\left\|\frac{2^{j\alpha}|f\chi_j|}{\beta_1}\right\|_{L^{p_1(\cdot)q_1(\cdot)}} \leq 1$, we deduce that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum\limits_{k+2}^{\infty}S_\gamma(f_j)\chi_k|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\leq \sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}|\sum\limits_{j=k+2}^{\infty}S_\gamma(f_j)\chi_k|}{\beta_1}\right\|_{L^{p_1(\cdot)}}^{(q_2^3)_k}$$

$$C \leq \sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-jn}\left\|\frac{f_j}{\beta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right]^{(q_2^3)_k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-jn}\left\|\frac{f_j}{\beta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}}|B_j|\right]^{(q_2^3)_k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha+n\delta_{12})}\left\|\left(\frac{2^{j\alpha}f\chi_j}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}^{\frac{1}{(q_1)_+}}\right]^{(q_2^3)_k}$$

where

$$(q_2^3)_k = \begin{cases} (q_2)_- & \left\|\left(\frac{2^{k\alpha}|\sum\limits_{j=k+2}^{\infty}S_\gamma(f_j)\chi_k|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq 1, \\ (q_2)_+ & \left\|\left(\frac{2^{k\alpha}|\sum\limits_{j=k+2}^{\infty}S_\gamma(f_j)\chi_k|}{\beta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

Notice that $(q_2)_- \geq (q_1)_+$ and $\alpha > -n\delta_{12}$, proceeding as in the estimate of β_{11}, we get that

$$\beta_{13} \leq C\beta_1 \leq C\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 3.3. □

4. BMO estimate for the commutator of intrinsic square functions

Let $b \in BMO(\mathbb{R}^n)$. Wang in [5] obtained some boundedness results for the commutator $[b, S_\gamma]$ in the framework of weighted Morrey spaces.

Lemma 4.1. *Let $1 < p < \infty$, $0 < \beta \leq 1$ and $w \in A_p$. Suppose that $b \in BMO(\mathbb{R}^n)$, then there exists a constant $C > 0$ independent of f such that*

$$\|[b, S_\gamma](f)\|_{L^p_w} \leq C\|f\|_{L^p_w}.$$

We can easy apply Lemma 4.1 and Lemma 3.2 to get the boundedness of the commutator $[b, S_\gamma]$ in $L^{p(\cdot)}$.

Theorem 4.2. *Let $b \in BMO(\mathbb{R}^n)$. Suppose that $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \geq (q_1)_+$ and $0 < \gamma \leq 1$. If $-n\delta_{12} < \alpha < n\delta_{11}$, where δ_{11}, δ_{12} are the constants in Lemma 2.5, then the operator $[b, S_\gamma]$ is bounded from $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n\right)$ to $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ $\left(K^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n\right)$.*

Proof. Let $b \in BMO(\mathbb{R}^n)$, $f \in \dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$. Let us decompose f as follows:

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

By the definition of the norm in $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have

$$\|[b,S_\gamma](f)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|[b,S_\gamma](f)\chi_k|}{\beta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq 1\right\}.$$

Since

$$\left\|\left(\frac{2^{k\alpha}|[b,S_\gamma](f)(f)\chi_k|}{\beta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{\infty}[b,S_\gamma](f_j)\chi_k|}{\beta_{21}+\beta_{22}+\beta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b,S_\gamma](f_j)\chi_k|}{\beta_{21}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} + C\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-1}^{k+1}[b,S_\gamma](f_j)\chi_k|}{\beta_{22}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$+C\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b,S_\gamma](f_j)\chi_k|}{\beta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}},$$

where

$$\beta_{21} = \left\|\left\{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b,S_\gamma](f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

$$\beta_{22} = \left\|\left\{2^{k\alpha}|\sum_{j=k-1}^{k+1}[b,S_\gamma](f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

$$\beta_{23} = \left\|\left\{2^{k\alpha}|\sum_{j=k+2}^{\infty}[b,S_\gamma](f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})}.$$

If $\beta = \beta_{21} + \beta_{22} + \beta_{23}$ thus

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|[b,S_\gamma](f_j)\chi_k|}{\beta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C.$$

Then

$$\|[b,S_\gamma](f)\chi_k\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} \leq C\beta_1 \leq C[\beta_{21}+\beta_{22}+\beta_{23}].$$

Hence, once we prove that

$$\beta_{21} \leq C\|b\|_*\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}, \quad \beta_{22} \leq C\|b\|_*\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}, \quad \beta_{23} \leq C\|b\|_*\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)},$$

we are done. Let us set $\beta_1 = \|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$.

For β_{22}, by the boundedness of $[b, S_\gamma]$ on $L^{p(\cdot)}$, and using an argument similar to that in the estimate for β_{12}, it follows that

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-1}^{k+1} [b, S_\gamma](f_j)\chi_k|}{\beta_1 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C,$$

which implies that

$$\beta_{22} \leq C\beta_1 \|b\|_* \leq C\|b\|_* \|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Now let us deal with the estimate for β_{21}. Let $x \in C_k$, $j \leq k-2$. By the estimate of $S_\gamma(f_j)(x)$ in the proof of Theorem 3.3, we have

$$S_\gamma(f_j)(x) \leq C2^{-kn} \|f_j\|_{L^1(\mathbb{R}^n)}.$$

From this inequality, we obtain that

$$[b, S_\gamma](f_j)(x) = |S_\gamma[(b(x) - b)f_j](x)| \leq C2^{-kn} \|(b(\cdot) - b)f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, using Lemma 2.6, we have

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-2} [b, S_\gamma](f_j)\chi_k|}{\beta_1 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\
&\leq C \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} |\sum_{j=-\infty}^{k-2} |x|^{-n} \|(b(\cdot) - b)f_j\|_{L^1(\mathbb{R}^n)} \chi_k|}{\beta_1 \|b\|_*} \right\|_{L^{p_1(\cdot)}}^{(q_2^2)_k} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-2} \left\| \frac{|(b - b_j)f_j|}{\beta_1 \|b\|_*} \right\|_{L^1(\mathbb{R}^n)} \|\chi_k\|_{L^{p_1(\cdot)}} \right)^{(q_2^2)_k} \\
&+ C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-2} \left\| \frac{|f_j|}{\beta_1} \right\|_{L^1(\mathbb{R}^n)} \frac{1}{\|b\|_*} \|(b - b_j)\chi_k\|_{L^{p_1(\cdot)}} \right)^{(q_2^2)_k},
\end{aligned}$$

where

$$(q_2^2)_k = \begin{cases} (q_2)_- & \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-2} [b,S_\gamma](f_j)\chi_k|}{\beta_1 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq 1, \\ (q_2)_+ & \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-2} [b,S_\gamma](f_j)\chi_k|}{\beta_1 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

Applying the generalized Hölder's inequality (Lemma 2.3) and Lemmas 2.4, 2.5, 2.7, we get that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum\limits_{j=-\infty}^{k-2}[b,S_\gamma](f_j)\chi_k|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\sum_{k=-\infty}^{k_\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-2}\left\|\frac{|f_j|}{\beta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\frac{|(b-b_j)\chi_{B_j}|}{\|b\|_*}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right)^{(q_2^2)_k}$$

$$+C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-2}\left\|\frac{|f_j|}{\beta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}(k-j)\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right)^{(q_2^2)_k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left(2^{k(\alpha-n)}\sum_{j=-\infty}^{k-2}(k-j)\left\|\frac{|f_j|}{\beta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}|B_k|\right)^{(q_2^2)_k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}(k-j)2^{(k-j)(\alpha-n\delta_{11})}\left\|\left(\frac{|f\chi_j|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}}\right)^{(q_2^2)_k}.$$

Furthermore, using an argument similar to that in the estimate for β_{11} , we have

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum\limits_{j=-\infty}^{k-2}[b,S_\gamma](f_j)\chi_k|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\left[\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{\alpha j}f\chi_j|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{\infty}(k-j)2^{(k-j)(\alpha-n\delta_{11})}\right]^{q_*}\leq C,$$

here $q_*=\min\limits_{k\in\mathbb{N}}\frac{(q_2^2)_k}{(q_1)_+}$.
This implies that

$$\beta_{21}\leq C\beta_1\|b\|_*\leq C\|b\|_*\|f\|_{\dot{K}^{\alpha,q_1(\cdot),}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Finally, we estimate β_{23}. Let $x\in C_k$, $j\geq k+2$. By the estimation of $S_\gamma(f_j)(x)$ in the proof of Theorem 3.3, we have

$$S_\gamma(f_j)(x)\leq C2^{-jn}\|f_j\|_{L^1(\mathbb{R}^n)}.$$

From the above inequality, we obtain that

$$[b,S_\gamma](f_j)(x)=|S_\gamma[(b(x)-b)f_j](x)|\leq C2^{-jn}\|(b(\cdot)-b)f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, when $\alpha > -n\delta_{12}$, proceeding as in the estimate of β_{21}, we get that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum\limits_{j=k+2}^{\infty}[b,S_\gamma](f_j)\chi_k|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}|\sum\limits_{j=k+2}^{\infty}2^{-jn}\|(b(\cdot)-b)f_j\|_{L^1(\mathbb{R}^n)}\chi_k|}{\beta_1\|b\|_*}\right\|_{L^{p_1(\cdot)}}^{(q_2^3)_k}$$

$$\leq C\sum_{k=-\infty}^{k_\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-jn}\left\|\frac{|(b-b_k)f_j|}{\beta_1\|b\|_*}\right\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{p_1(\cdot)}}\right)^{(q_2^3)_k}$$

$$+C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-jn}\left\|\frac{|f_j|}{\beta_1}\right\|_{L^1(\mathbb{R}^n)}\frac{1}{\|b\|_*}\|(b-b_k)\chi_k\|_{L^{p_1(\cdot)}}\right)^{(q_2^3)_k}$$

$$\leq C\sum_{k=0}^{k_0}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-jn}(j-k)|B_j|\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}}\left\|\frac{|f_j|}{\beta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^3)_k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}(j-k)2^{(k-j)(\alpha+n\delta_{12})}\left\|\left(\frac{|f\chi_j|}{\beta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}}\right)^{(q_2^3)_k}\leq C,$$

where

$$(q_2^3)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k+2}^{\infty}[b,S_\gamma](f_j)\chi_k|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k+2}^{\infty}[b,S_\gamma](f_j)\chi_k|}{\beta_1\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}> 1.\end{cases}$$

The above calculations imply that

$$\beta_{23}\leq C\beta_1\|b\|_*\leq C\|b\|_*\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 4.2. □

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