Functional Analysis FAC – SEAMS
Volume 1, 2022, ID 6

On Beurling Spaces Provided With a Weight Dependent Convolution

Abudulaï Issa [1] and Yaogan Mensah [1,2] ✉

[1] *Department of Mathematics, University of Lomé, 1 BP 1515 Lomé 01, Lomé, Togo*

[2] *ICMPA-Unesco Chair, University of Abomey-Calavi, Cotonou, Benin*

ABSTRACT. In this paper, we prove the stability of Beurling spaces under the action of a generalized translation operator. Also, some interesting properties of Beurling spaces endowed with a weight dependent convolution are studied. We obtain, among other results, a sufficient condition on the weight so that some Beurling spaces can inherit a Banach algebra structure.

1. Introduction and preliminaries

Convolution product is a useful tool in many areas of Mathematics and applied sciences. It appears for instance in Statistics, Signal processing, Harmonic analysis, etc. For complex integrable functions f and g defined on a locally compact group G endowed with a left Haar measure, the convolution product is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy. \tag{1.1}$$

Let us consider the Beurling spaces (see [1, 2, 7])

$$L^p_\omega(G) = \left\{ f : G \to \mathbb{C} \, : \, \int_G |f(x)|^p \omega(x)dx < \infty \right\}, 1 \leqslant p < +\infty, \tag{1.2}$$

where ω is a Beurling weight, that is, ω is a nonnegative continuous functions defined on G such that $\omega(xy) \leq \omega(x)\omega(y)$ and $\omega(e) = 1$, here e is the neutral element of the group G.

The space $L^p_\omega(G)$ is provided with the norm

$$\|f\|_{p,\omega} = \left(\int_G |f(x)|^p \omega(x)dx \right)^{\frac{1}{p}}. \tag{1.3}$$

In [6], a generalization of the convolution product was introduced by Mahmoodi by the formula

$$f *_\omega g(x) = \int_G f(y)g(y^{-1}x) \frac{\omega(y)\omega(y^{-1}x)}{\omega(x)} dy. \tag{1.4}$$

Date submitted: 2022-08-24.
2020 *Mathematics Subject Classification.* Primary 43A15; Secondary 43A20, 22D15.
Key words and phrases. Weight, Beurling space, Convolution, Banach algebra.

One recovers the classical convolution when $\omega \equiv 1$. Throughout this paper, we set

$$\mathcal{L}^p_\omega(G) = (L^p_\omega(G), \|\cdot\|_{p,\omega}, *_\omega),\ p \geq 1.$$

In [4], Issa and Mensah characterized multipliers of $\mathcal{L}^1_\omega(G)$ by the means of a generalized translation operator Γ^s_ω the properties of which will be study in Section 2.

The aim of this paper is to pursue the work started in [4] by searching properties (categorical or not) of the spaces $\mathcal{L}^p_\omega(G)$.

The rest of the paper is organized as follows. Section 2 contains results about a generalized translation operators Γ^s_ω and Section 3 is devoted to the properties of the weighted convolution product in the Beurling spaces $\mathcal{L}^p_\omega(G)$.

2. Properties of the Γ^s_ω operators

We start this section with a density result which is interested on its own.

Theorem 2.1. *The space $\mathcal{L}^1_\omega(G)$ is a dense subspace of $(L^1(G), \|\cdot\|_1)$.*

Proof. We denote by $\mathcal{C}_c(G)$ the space of continuous compactly supported complex functions on G. This space is known to be dense in $L^1(G)$. Let $f \in \mathcal{L}^1_\omega(G)$. Then one has

$$\int_G |f(x)|\omega(x)dx < \infty.$$

The hypothesis $1 \leqslant \omega$ implies $\int_G |f(x)|dx \leqslant \int_G |f(x)|\omega(x)dx$. Thus $\mathcal{L}^1_\omega(G) \subset L^1(G)$. Also, if $f \in \mathcal{C}_c(G)$ then $f\omega \in \mathcal{C}_c(G)$. Therefore $f \in \mathcal{L}^1_\omega(G)$ by the inclusion $\mathcal{C}_c(G) \subset L^1(G)$. Hence $\mathcal{C}_c(G) \subset \mathcal{L}^1_\omega(G)$. We have

$$\mathcal{C}_c(G) \subset \mathcal{L}^1_\omega(G) \subset L^1(G).$$

Finally, taking the closures with respect to the topology of $L^1(G)$, we obtain $\overline{\mathcal{C}_c}(G) \subset \overline{\mathcal{L}^1_\omega(G)} \subset L^1(G)$. This implies $\overline{\mathcal{L}^1_\omega(G)} = L^1(G)$ since $\mathcal{C}_c(G)$ is dense in $L^1(G)$. Hence The space $\mathcal{L}^1_\omega(G)$ is a dense subspace of $(L^1(G), \|\cdot\|_1)$. □

We consider the mulitplication operateur M_ω defined by

$$(M_\omega f)(x) = \omega(x)f(x),\ f \in \mathcal{L}^1_\omega(G) \tag{2.1}$$

and the translation operator $\tau_s,\ s \in G$ by

$$(\tau_s f)(x) = f(s^{-1}x),\ f \in \mathcal{L}^1_\omega(G). \tag{2.2}$$

Now, set

$$\Gamma^s_\omega f(x) = \frac{\tau_s M_\omega f(x)}{\omega(x)},\ f \in \mathcal{L}^1_\omega(G). \tag{2.3}$$

The operator $\Gamma^s_\omega = \tau_s$ reduced to the translation operator τ_s if $\omega \equiv 1$. We may need the following result.

Theorem 2.2. *([4, Proposition 3.1]) If $f \in \mathcal{L}^1_\omega(G)$, then $\|\Gamma^s_\omega f\|_{1,\omega} = \|f\|_{1,\omega}$.*

Theorem 2.3. *Let $f \in \mathcal{L}^p_\omega(G)$ and $s \in G$. Then*

$$[\omega(s)]^{\frac{1-p}{p}} \|f\|_{p,\omega} \leqslant \|\Gamma^s_\omega f\|_{p,\omega} \leqslant \left[\omega(s^{-1})\right]^{\frac{p-1}{p}} \|f\|_{p,\omega}. \tag{2.4}$$

Proof. Let $f \in \mathcal{L}^p_\omega(G)$. For all $s \in G$, one has

$$\begin{aligned}
\|\Gamma^s_\omega f\|^p_{p,\omega} &= \int_G |\Gamma^s_\omega f(x)|^p \omega(x)dx \\
&= \int_G |f(s^{-1}x)|^p \left(\frac{\omega(s^{-1}x)}{\omega(x)}\right)^p \omega(x)dx \\
&= \int_G |f(x)|^p \left(\frac{\omega(x)}{\omega(sx)}\right)^p \omega(x)dx \\
&= \int_G |f(x)|^p \left(\frac{\omega(x)}{\omega(sx)}\right)^p \omega(sx)dx \\
&= \int_G |f(x)|^p \left(\frac{\omega(x)}{\omega(sx)}\right)^{p-1} \omega(x)dx.
\end{aligned}$$

However $\omega(x) = \omega(s^{-1}sx) \leqslant \omega(s^{-1})\omega(sx)$. Therefore $\dfrac{\omega(x)}{\omega(sx)} \leqslant \omega(s^{-1})$. Then

$$\begin{aligned}
\|\Gamma^s_\omega f\|^p_{p,\omega} &\leqslant \int_G |f(x)|^p \left[\omega(s^{-1})\right]^{p-1} \omega(x)dx \\
&\leqslant \left[\omega(s^{-1})\right]^{p-1} \int_G |f(x)|^p \omega(x)dx = \left[\omega(s^{-1})\right]^{p-1} \|f\|^p_{p,\omega}.
\end{aligned}$$

Thus $\|\Gamma^s_\omega f\|_{p,\omega} \leqslant [\omega(s^{-1})]^{\frac{p-1}{p}} \|f\|_{p,\omega}$.

On the other hand,

$$\begin{aligned}
\|\Gamma^s_\omega f\|^p_{p,\omega} &= \int_G |\Gamma^s_\omega f(x)|^p \omega(x)dx \\
&= \int_G |f(s^{-1}x)|^p \left(\frac{\omega(s^{-1}x)}{\omega(x)}\right)^p \omega(x)dx \\
&= \int_G |f(s^{-1}x)|^p \left(\frac{\omega(s^{-1}x)}{\omega(x)}\right)^{p-1} \omega(s^{-1}x)dx
\end{aligned}$$

However $\omega(s)\omega(s^{-1}x) \geqslant \omega(x)$. Therefore $\dfrac{\omega(s^{-1}x)}{\omega(x)} \geqslant [\omega(s)]^{-1}$. Then

$$\begin{aligned}
\|\Gamma^s_\omega f\|^p_{p,\omega} &\geqslant \int_G |f(s^{-1}x)|^p [\omega(s)]^{1-p} \omega(s^{-1}x)dx \\
&\geqslant [\omega(s)]^{1-p} \int_G |f(s^{-1}x)|^p \omega(s^{-1}x)dx \\
&\geqslant [\omega(s)]^{1-p} \|f\|^p_{p,\omega}.
\end{aligned}$$

Thus

$$\|\Gamma^s_\omega f\|_{p,\omega} \geqslant [\omega(s)]^{\frac{1-p}{p}} \|f\|_{p,\omega}.$$

Finally

$$[\omega(s)]^{\frac{1-p}{p}}\|f\|_{p,\omega}\leqslant\|\Gamma_\omega^s f\|_{p,\omega}\leqslant\left[\omega(s^{-1})\right]^{\frac{p-1}{p}}\|f\|_{p,\omega}.$$

□

As a consequence of the above theorem, we deduce respectively the stability of $\mathcal{L}^p_\omega(G)$ under the action of the operator Γ^s_ω and the uniform continuity of Γ^s_ω.

Corollary 2.4. *$f\in\mathcal{L}^p_\omega(G)$ if and only of $\Gamma^s_\omega f\in\mathcal{L}^p_\omega(G)$.*

Proof. From Theorem 2.3, we deduce that $\|f\|_{p,\omega}<\infty$ if and only if $\|\Gamma^s_\omega f\|_{p,\omega}<\infty$. □

Corollary 2.5. *Fix $s\in G$. The map $f\longmapsto\Gamma^s_\omega f$ is uniformly continuous from $\mathcal{L}^p_\omega(G)$ into $\mathcal{L}^p_\omega(G)$.*

Proof. Let $f,g\in\mathcal{L}^p_\omega(G)$. From Theorem 2.3, we have

$$\|\Gamma^s_\omega f-\Gamma^s_\omega g\|_{p,\omega}=\|\Gamma^s_\omega(f-g)\|_{p,\omega}\leqslant C\|f-g\|_{p,\omega}.$$

where $C=[\omega(s^{-1})]^{\frac{p-1}{p}}$. Thus the map $f\longmapsto\Gamma^s_\omega f$ is uniformly continuous. □

3. The weight dependent convolution

Let us recall that the weight dependent convolution is defined by

$$f*_\omega g(x)=\int_G f(y)g(y^{-1}x)\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}dy, \tag{3.1}$$

under the assumption that the latter integral exists. In the following theorem, we provide a sufficient condition under which the space $\mathcal{L}^p_\omega(G)$ is a Banach algebra.

Theorem 3.1. *Let $p>1$. The space $\mathcal{L}^p_\omega(G)$ is a Banach algebra if $\omega*\omega\leqslant\omega$.*

Proof. The space Beurling space $(L^p_\omega(G),\ \|\cdot\|_{p,\omega})$ is known to be a Banach space. So it suffices to show that

$$\|f*_\omega g\|_{p,\omega}\leqslant\|f\|_{p,\omega}\|g\|_{p,\omega},\ \forall f,g\in\mathcal{L}^p_\omega(G)$$

under the hypothesis $\omega*\omega\leqslant\omega$. Let $f,g\in\mathcal{L}^p_\omega(G)$.

$$\begin{aligned}
f*_\omega g(x) &= \int_G f(y)g(y^{-1}x)\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}dy\\
&= \int_G f(y)g(y^{-1}x)\left[\omega(y^{-1}x)\omega(y)\right]^{\frac{1}{p}}\frac{1}{\omega(x)}\left[\omega(y^{-1}x)\omega(y)\right]^{\frac{p-1}{p}}dy.
\end{aligned}$$

Now we use the Hölder's inequality to get

$$\begin{aligned}
|f*_\omega g(x)| &\leqslant \left(\int_G|f(y)|^p\omega(y)|g(y^{-1}x)|^p\omega(y^{-1}x)dy\right)^{\frac{1}{p}}\left(\int_G\frac{1}{(\omega(x))^{\frac{p}{p-1}}}(\omega(y^{-1}x)\omega(y))dy\right)^{\frac{p-1}{p}}\\
&\leqslant \left(\int_G|f(y)|^p\omega(y)|g(y^{-1}x)|^p\omega(y^{-1}x)dy\right)^{\frac{1}{p}}\frac{1}{\omega(x)}\left(\int_G\left(\omega(y^{-1}x)\omega(y)\right)dy\right)^{\frac{p-1}{p}}\\
&\leqslant \left(\int_G|f(y)|^p\omega(y)|g(y^{-1}x)|^p\omega(y^{-1}x)dy\right)^{\frac{1}{p}}\frac{1}{\omega(x)}\left[W(x)\right]^{\frac{p-1}{p}}
\end{aligned}$$

where we have set $W = \omega * \omega$. It follows that

$$\begin{aligned}\int_G |(f *_\omega g)(x)|^p (\omega(x))^p W^{1-p}(x)dx &\leqslant \left(\int_G |f(y)|^p \omega(y)dy\right)\left(\int_G |g(y^{-1}x)|^p \omega(y^{-1}x)dx\right)\\ &\leqslant \|f\|_{p,\omega}^p \|g\|_{p,\omega}^p.\end{aligned}$$

Moreover,

$$\omega * \omega \leqslant \omega \Longleftrightarrow W \leqslant \omega \Longrightarrow W^{1-p} \geqslant w^{1-p} \Longrightarrow \omega^p W^{1-p} \geqslant \omega.$$

Hence

$$\int_G |(f *_\omega g)(x)|^p \omega(x)dx \leqslant \int_G |(f *_\omega g)(x)|^p (\omega(x))^p W^{1-p}(x)dx.$$

Thus

$$\|f *_\omega g\|_{p,\omega} \leqslant \|f\|_{p,\omega}\|g\|_{p,\omega}.$$

□

Remark 3.2. The fact that $\mathcal{L}^1_\omega(G)$ is a Banach algebra was proved in [6].

In what follows, we set $\check{\omega}(y) = \omega(y^{-1}),\ y \in G$.

Theorem 3.3. *Let $p > 1$. If $f \in \mathcal{L}^p_\omega(G)$, $g \in \mathcal{L}^1_\omega(G)$ and $g\check{\omega} \in \mathcal{L}^1_\omega(G)$ then $f *_\omega g \in \mathcal{L}^p_\omega(G)$ and*

$$\|f *_\omega g\|_{p,\omega} \leq \|g\check{\omega}\|_{1,\omega}^{1-\frac{1}{p}} \|g\|_{1,\omega}^{\frac{1}{p}} \|f\|_{p,\omega}. \tag{3.2}$$

Proof. Let $p > 1$ and let q be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

$$\begin{aligned}|(f *_\omega g)(x)| &\leqslant \int_G |g(y^{-1}x)||f(y)|\frac{\omega(y)\omega(y^{-1}x)}{\omega(x)}dy\\ &= \int_G |\Gamma^y_\omega g(x)|\omega(y)|f(y)|dy\\ &= \int_G |\Gamma^y_\omega g(x)\omega(y)|^{\frac{1}{p}}|f(y)||\Gamma^y_\omega g(x)\omega(y)|^{\frac{1}{q}}dy\\ &\leqslant \left(\int_G |\Gamma^y_\omega g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}}\left(\int_G |\Gamma^y_\omega g(x)\omega(y)dy|\right)^{\frac{1}{q}}\\ &= \left(\int_G |\Gamma^y_\omega g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}}\left(\int_G |\frac{g(y^{-1}x)\omega(y^{-1}x)}{\omega(x)}\omega(y)|dy\right)^{\frac{1}{q}}\\ &= \left(\int_G |\Gamma^y_\omega g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}}\left(\int_G |\frac{g(y)\omega(y)}{\omega(x)}\omega(xy^{-1})|dy\right)^{\frac{1}{q}}\\ &\leqslant \left(\int_G |\Gamma^y_\omega g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}}\left(\int_G |g(y)\omega(y)\omega(y^{-1})|dy\right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_G |\Gamma_\omega^y g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}} \left(\int_G |g(y)\omega(y^{-1})\omega(y)|dy\right)^{\frac{1}{q}} \\
&= \left(\int_G |\Gamma_\omega^y g(x)\omega(y)||f(y)|^p dy\right)^{\frac{1}{p}} \|g\check{\omega}\|_{1,\omega}^{\frac{1}{q}} \\
&= ((|g| *_\omega |f|^p)(x))^{\frac{1}{p}} \|g\check{\omega}\|_{1,\omega}^{\frac{1}{q}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_G |(f *_\omega g)(x)|^p \omega(x)dx &\leqslant \int_G (|g| *_\omega |f|^p)(x) \|g\check{\omega}\|_{1,\omega}^{\frac{p}{q}} \omega(x)dx \\
&= \|g\check{\omega}\|_{1,\omega}^{\frac{p}{q}} \int_G (|g| *_\omega |f|^p)(x)\omega(x)dx \\
&\leqslant \|g\check{\omega}\|_{1,\omega}^{\frac{p}{q}} \|g\|_{1,\omega} \||f|^p\|_{1,\omega} \\
&= \|g\check{\omega}\|_{1,\omega}^{\frac{p}{q}} \|g\|_{1,\omega} \|f\|_{p,\omega}^p.
\end{aligned}
$$

Hence

$$
\|f *_\omega g\|_{p,\omega} \le \|g\check{\omega}\|_{1,\omega}^{1-\frac{1}{p}} \|g\|_{1,\omega}^{\frac{1}{p}} \|f\|_{p,\omega}.
$$

□

Remark 3.4. If $\omega \equiv 1$ then $\check{\omega} \equiv 1$ and we recover from the inequality (3.2) the well-known classical inequality

$$
\|f * g\|_p \le \|g\|_1 \|f\|_p \tag{3.3}
$$

for $f \in L^p(G)$ and $g \in L^1(G)$.

We recall the following fact which we may use in the next theorem.

Theorem 3.5. *([3, Corollary 12.5]) Let $f_1, f_2, ..., f_n$ be nonnegative functions in $L^1(G)$ and let $\alpha_1, \alpha_2, ..., \alpha_n$ be positive numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Then $f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_n^{\alpha_n} \in L^1(G)$ and*

$$
\int_G f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_n^{\alpha_n} dt \leqslant \|f_1\|_1^{\alpha_1} \|f_2\|_1^{\alpha_2} \cdots \|f_n\|_1^{\alpha_n}.
$$

Theorem 3.6. *Let p and q be real numbers such that $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} > 1$, and let $r = \frac{pq}{p+q-pq}$, so that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Let $f \in \mathcal{L}^p_{\omega^p}(G)$ and $g \in \mathcal{L}^q_{\omega^q}(G)$. Then $f *_\omega g \in L^r(G)$ and*

$$
\|f *_\omega g\|_r \leqslant \|f\|_{p,\omega^p} \|g\|_{q,\omega^q}.
$$

Proof.

$$
\begin{aligned}
|(f *_\omega g)(x)| &\leqslant \int_G |f(y^{-1}x)g(y)\frac{\omega(y^{-1}x)\omega(y)}{\omega(x)}|dy \\
&\leqslant \int_G |f(y^{-1}x)g(y)\omega(y^{-1}x)\omega(y)|dy \\
&= \int_G \left[|f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|\right]^{\frac{1}{r}} |g(y)\omega(y)|^{1-\frac{q}{r}} |f(y^{-1}x)\omega(y^{-1}x)|^{1-\frac{p}{r}} dy \\
&= \int_G \left[|f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|\right]^{\frac{1}{r}} |g^q(y)\omega^q(y)|^{\frac{1}{q}-\frac{1}{r}} |f^p(y^{-1}x)\omega^p(y^{-1}x)|^{\frac{1}{p}-\frac{1}{r}} dy.
\end{aligned}
$$

Since $\frac{1}{p}-\frac{1}{r}=\frac{q-1}{q}$ and $\frac{1}{q}-\frac{1}{r}=\frac{p-1}{p}$, by applying Theorem 3.5, one has

$$\int_G |f(y^{-1}x)g(y)\frac{\omega(y^{-1}x)\omega(y)}{\omega(x)}|dy \leqslant \left[|\int_G f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|dy\right]^{\frac{1}{r}}\left(\int_G |g^q(y)\omega^q(y)|dy\right)^{\frac{p-1}{p}} \left(\int_G |f^p(y^{-1}x)\omega^p(y^{-1}x)|dy\right)^{\frac{q-1}{q}}.$$

It follows that

$$|(f*_\omega g)(x)|^r \leqslant (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\int_G |f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|dy.$$

Therefore

$$\begin{aligned}
&\int_G |(f*_\omega g)(x)|^r dx\\
\leqslant\ & (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\int_G\int_G \left[|f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|\right]dydx\\
\leqslant\ & (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\left[\int_G\left(\int_G |f(y^{-1}x)|^p\omega^p(y^{-1}x)dx\right)|g(y)|^q\omega^q(y)dy\right]\\
=\ & (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}(\|f\|_{p,\omega^p})^p\left[\int_G g^q(y)\omega^q(y)|dx\right]\\
=\ & (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}(\|f\|_{p,\omega^p})^p(\|g\|_{q,\omega^q})^q\\
\leqslant\ & (\|f\|_{p,\omega^p})^{p+\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{q+\frac{qr(p-1)}{p}}=\|f\|^r_{p,\omega^p}\|g\|^r_{q,\omega^q}.
\end{aligned}$$

Thus

$$\|f*_\omega g\|_r \leqslant \|f\|_{p,\omega^p}\|g\|_{q,\omega^q}.$$

□

Theorem 3.7. *Let p and q be real numbers such that $1<p<\infty$, $1<q<\infty$, $\frac{1}{p}+\frac{1}{q}>1$, and let $r=\frac{pq}{p+q-pq}$, so that $\frac{1}{p}+\frac{1}{q}-\frac{1}{r}=1$. Let $f\in\mathcal{L}^p_{\omega^{p+1}}(G)$ and $g\in\mathcal{L}^q_{\omega^{q+1}}(G)$. Then $f*_\omega g\in\mathcal{L}^r_\omega(G)$ and*

$$\|f*_\omega g\|_{r,\omega}\leqslant\|f\|_{p,\omega^{p+1}}\|g\|_{q,\omega^{q+1}}.$$

Proof. In the proof of Theorem 3.6 we established that

$$|(f*_\omega g)(x)|^r \leqslant (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\int_G |f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|dy.$$

Set $A=\int_G |(f*_\omega g)(x)|^r\omega(x)dx$. We have

$$\begin{aligned}
A\ \leqslant\ & (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\int_G\int_G \left[|f^p(y^{-1}x)g^q(y)\omega^p(y^{-1}x)\omega^q(y)|\right]dy\omega(x)dx\\
\leqslant\ & (\|f\|_{p,\omega^p})^{\frac{pr(q-1)}{q}}(\|g\|_{q,\omega^q})^{\frac{qr(p-1)}{p}}\left[\int_G\left(\int_G |f(y^{-1}x)|^p\omega^p(y^{-1}x)\omega(x)dx\right)|g(y)|^q\omega^q(y)dy\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(\|f\|_{p,\omega^p}\right)^{\frac{pr(q-1)}{q}} \left(\|g\|_{q,\omega^q}\right)^{\frac{qr(p-1)}{p}} \left[\int_G \left(\int_G |f(x)|^p \omega^p(x)\omega(yx)dx\right) |g(y)|^q \omega^q(y)dy\right] \\
&\leqslant \left(\|f\|_{p,\omega^p}\right)^{\frac{pr(q-1)}{q}} \left(\|g\|_{q,\omega^q}\right)^{\frac{qr(p-1)}{p}} \left[\int_G \left(\int_G |f(x)|^p \omega^{p+1}(x)dx\right) |g(y)|^q \omega^{q+1}(y)dy\right] \\
&= \left(\|f\|_{p,\omega^p}\right)^{\frac{pr(q-1)}{q}} \left(\|g\|_{q,\omega^q}\right)^{\frac{qr(p-1)}{p}} \left(\|f\|_{p,\omega^{p+1}}\right)^p \left[\int_G g^q(y)\omega^{q+1}(y)|dx\right] \\
&= \left(\|f\|_{p,\omega^p}\right)^{\frac{pr(q-1)}{q}} \left(\|g\|_{q,\omega^q}\right)^{\frac{qr(p-1)}{p}} \left(\|f\|_{p,\omega^{p+1}}\right)^p \left(\|g\|_{q,\omega^{q+1}}\right)^q \\
&\leqslant \left(\|f\|_{p,\omega^{p+1}}\right)^{p+\frac{pr(q-1)}{q}} \left(\|g\|_{q,\omega^{q+1}}\right)^{q+\frac{qr(p-1)}{p}} = \|f\|^r_{p,\omega^{p+1}} \|g\|^r_{q,\omega^{q+1}}
\end{aligned}
$$

because $\|f\|_{p,\omega^p} \leqslant \|f\|_{p,\omega^{p+1}}$ for $f \in \mathcal{L}^p_{\omega^{p+1}}(G)$ and $\|g\|_{q,\omega^q} \leqslant \|g\|_{q,\omega^{q+1}}$ for $g \in \mathcal{L}^q_{\omega^{q+1}}(G)$. Thus $\|f *_\omega g\|_{r,\omega} \leqslant \|f\|_{p,\omega^{p+1}} \|g\|_{q,\omega^{q+1}}$. □

References

[1] Benazzouz, A. (1986). Contribution à l'analyse harmonique des algèbres de Beurling généralisées, Thèse de 3ème cycle, Faculté des Sciences de Rabat.

[2] Bourouihiya, A. (2006). Beurling weighted spaces, product-convolution operators and the tensor product of frames, PhD thesis, University of Maryland, USA.

[3] Hewitt, E., Ross, K. A. (1979). *Abstract Harmonic Analysis I*, Springer-Verlag,.

[4] Mensah, Y., Issa, A. (2020). Multipliers on weighted group algebras. Gulf Journal of Mathematics, 8(2), 35-45.

[5] EL Kinani, A., Roukbi, A., Benazzouz, A. (2009). Structure d'algèbre de Banach sur l'espace à poids $L^p_\omega(G)$, Le Mathematiche, vol. LXIV, Fasc. I, 179-193.

[6] Mahmoodi, A. (2009). A new convolution on Beurling algebras, Mathematical Sciences, 3, 63-82.

[7] Reiter, H., Stegeman, J. D. (2000). Classical harmonic analysis and locally compact groups, London Math. Society Monographs, vol. 22, Clarendon Press, Oxford.