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Continuous Controlled K-G-Frames for Hilbert C^*-Module

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ABSTRACT. The purpose of this paper is the introduction and the study of the new concept that of continuous controlled K-g-Frame for Hilbert C^*-Modules which is a generalization of controlled K-g-Frames in Hilbert C*-Modules in discrete case. Also, we give some new properties.

1. Introduction and preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [8] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [6] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frame and Gabor frame [10]. Frames have been used in signal processing, image processing, data compression and sampling theory.

The concept of a generalization of frame to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [12] and independently by Ali, Antoine and Gazeau [1]. These frames are known as continuous frames. Gabardo and Han in [9] called them frames associated with measurable spaces, Askari-Hemmat, Dehghan and Radjabalipour in [3] called them generalized frames and in mathematical physics they are know as energy-staes.

In 2012, L. Gavruta [11] introduced the notion of K-frame in Hilbert space to study the atomic systems with respect to a bounded linear operator K. Controlled frames in Hilbert spaces have been introduced by P. Balazs [4] to improve the numerical efficiency of iterative algorithms for inverting the frame operator.

Controlled frames in C^*-modules were introduced by Rashidi and Rahimi [17], where the authors showed that they share many useful properties with their corresponding notions in a Hilbert spaces. Finally, we note that controlled K-g- frames in Hilbet spaces

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have been introduced by Dingli Hua and Yongdong Huang [13]. For more details, see [14–16, 19, 21, 23–27].

In this paper we introduce the notion of a continuous controlled K-g-frame in Hilbert C^*-modules.

In the following we briefly recall the definitions and basic properties of C^*-algebras and Hilbert $\mathcal{A}$-modules. Our references for C^*-algebras are [5, 7]. For a C^*-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of all positive elements of $\mathcal{A}$.

Definition 1.1. [20] Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.
(ii) $\langle ax + y, z \rangle_{\mathcal{A}} = a\langle x, z \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.
(iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x \rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We reserve the notation $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^*(\mathcal{H})$.

The following lemmas will be used to prove our mains results

Lemma 1.2. *[2]. Let $\mathcal{H}$ and $\mathcal{K}$ two Hilbert $\mathcal{A}$-modules and $T \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalente,*

(i) *T is surjective.*
(ii) *T^* is bounded below with respect to norm, i.e, there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$, $x \in \mathcal{K}$.*
(iii) *T^* is bounded below with respect to the inner product, i.e, there is $m' > 0$ such that,*

$$\langle T^*x, T^*x \rangle_{\mathcal{A}} \geq m' \langle x, x \rangle_{\mathcal{A}}, x \in \mathcal{K}$$

.

For the following theorem, $R(T)$ denote the range of the operattor T.

Theorem 1.3. *[28] Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$. Let $T, S \in End_{\mathcal{A}}^*(\mathcal{H})$. If $R(S)$ is closed, then the following statements are equivalent:*

(1) *$R(T) \subseteq R(S)$.*
(2) *$TT^* \leq \lambda^2 SS^*$ for some $\lambda \geq 0$.*
(3) *There exists $Q \in End_{\mathcal{A}}^*(\mathcal{H})$ such that $T = SQ$.*

2. Continuous controlled K-g-frames for Hilbert C^*-modules

Let X be a Banach space, (Ω, μ) a measure space, and $f : \Omega \to X$ a measurable function. Integral of the Banach-valued function f has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions. Since every C^*-algebra and Hilbert C^*-module is a Banach space thus we can use this integral and its properties.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules, $\{\mathcal{K}_w : w \in \Omega\}$ is a family of subspaces of $\mathcal{K}$, and $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_w)$ is the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ into $\mathcal{K}_w$. We define

$$l^2(\Omega, \{\mathcal{K}_w\}_{\omega\in\Omega}) = \left\{ x = \{x_w\}_{w\in\Omega} : x_w \in \mathcal{K}_w, \left\| \int_\Omega |x_w|^2 d\mu(w) \right\| < \infty \right\}.$$

For any $x = \{x_w : w \in \Omega\}$ and $y = \{y_w : w \in \Omega\}$, if the $\mathcal{A}$-valued inner product is defined by $\langle x, y\rangle = \int_\Omega \langle x_w, y_w\rangle_{\mathcal{A}} d\mu(w)$, the norm is defined by $\|x\| = \|\langle x, x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. The $l^2(\Omega, \{\mathcal{K}_w\}_{\omega\in\Omega})$ is a Hilbert C^*-module (see [18]).
Let $\mathcal{A}$ be a C^*-algebra, $l^2(\mathcal{A})$ is defined by,

$$l^2(\mathcal{A}) = \{\{a_\omega\}_{w\in\Omega} \subseteq \mathcal{A} : \| \int_\Omega a_\omega a^*_\omega d\mu(\omega)\| < \infty\}.$$

$l^2(\mathcal{A})$ is a Hilbert C^*-module with pointwise operations and the inner product defined by,

$$\langle \{a_\omega\}_{w\in\Omega}, \{b_\omega\}_{w\in\Omega}\rangle = \int_\Omega a_\omega b^*_\omega d\mu(\omega), \{a_\omega\}_{w\in\Omega}, \{b_\omega\}_{w\in\Omega} \in l^2(\mathcal{A}),$$

and,

$$\|\{a_\omega\}_{w\in\Omega}\| = (\int_\Omega a_\omega a^*_\omega d\mu(\omega))^{\frac{1}{2}}.$$

Let $GL^+(\mathcal{H})$ be the set of all positive bounded linear invertible operators on $\mathcal{H}$ with bounded inverse.

Definition 2.1. [14] Let $\Lambda = \{\Lambda_w\}_{w\in\Omega}$ be a family in $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_w)$ for all $\omega \in \Omega$, and $C, C' \in GL^+(\mathcal{H})$. We say that the family Λ is a (C, C')-controlled continuous g-frame for Hilbert C^*-module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$ if it is a continuous g-Bessel family and there is a pair of constants $0 < A, B$ such that, for any $f \in \mathcal{H}$,

$$A\langle f, f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_w C f, \Lambda_w C' f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathcal{A}} \quad . \tag{2.1}$$

A and B are called the (C, C')-controlled continuous g-frames bounds.

Definition 2.2. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a unital C^*-algebra, and $C, C' \in GL^+(\mathcal{H})$. A family of adjointable operators $\{\Lambda_\omega\}_{w\in\Omega} \subset End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_w)$ is said to be a continuous (C, C')-controlled K-g-frame for Hilbert C^*-module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$ if

- For all $f \in \mathcal{H}$, the function: $\omega \to \Lambda_\omega f$ is measurable.
- There exist two positive elements A and B such that

$$A\langle K^* f, K^* f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega C f, \Lambda_\omega C' f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathcal{A}}, f \in \mathcal{H}. \tag{2.2}$$

The elements A and B are called continuous (C, C')-controlled K-g-frame bounds.

If only the right-hand inequality of (2.2) is satisfied, we call a continuous (C, C')-controlled Bessel K-g-frame with Bessel bound B.

Example 2.3. Let $\mathcal{H} = \left\{ M = \begin{pmatrix} a & b & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix} / a, b, c, d \in \mathbb{C} \right\}$,
and $\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / a, b, c, d \in \mathbb{C} \right\}$
It's clair that $\mathcal{H}$ respectively $\mathcal{A}$ is a Hilbert space respectively a $\mathbb{C}^*$-algebra. Also it's known that $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.
Let C and C' be two operators respectively defined as follow,

$$\begin{aligned} C : \mathcal{H} &\longrightarrow \mathcal{H} \\ M &\longrightarrow \alpha M \end{aligned}$$

and

$$\begin{aligned} C' : \mathcal{H} &\longrightarrow \mathcal{H} \\ M &\longrightarrow \beta M \end{aligned}$$

where α and β are two reels numbers strictly greater than zero.
It's clair that $C, C' \in Gl^+(\mathcal{H})$.
Indeed, for each $M \in \mathcal{H}$ one has

$$C^{-1}(M) = \alpha^{-1} M \quad and \quad (C')^{-1}(M) = \beta^{-1} M.$$

Let $\Omega = [0, 1]$ endewed with the lebesgue's measure. It's clear that a measure space. Moreover, for $\omega \in \Omega$, we define the operator $\Lambda_w : \mathcal{H} \to \mathcal{H}$ by,

$$\Lambda_w(M) = w \begin{pmatrix} 0 & b & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix},$$

Λ_w is linear, bounded and selfadjoint.
In addition, for $M \in \mathcal{H}$, we have,

$$\begin{aligned} \int_\Omega \langle \Lambda_w C M, \Lambda_w C' M \rangle_{\mathcal{A}} d\mu(\omega) &= \int_\Omega \alpha\beta \begin{pmatrix} 0 & b & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ w\bar{b} & w\bar{c} \\ 0 & 0 \\ 0 & c \end{pmatrix} d\mu(\omega) \\ &= \int_\Omega \alpha\beta \begin{pmatrix} |b|^2 & b\bar{c} \\ c\bar{b} & |c|^2 \end{pmatrix} w^2 d\mu(\omega). \end{aligned}$$

It's clear that,

$$\begin{pmatrix} |b|^2 & b\bar{c} \\ c\bar{b} & |c|^2 \end{pmatrix} \leq \begin{pmatrix} |a|^2 + |b|^2 & b\bar{c} \\ c\bar{b} & |c|^2 + |d|^2 \end{pmatrix} = \|M\|_{\mathcal{A}}^2.$$

Then we have
$$\int_{\Omega}\langle \Lambda_w CM, \Lambda_w C^{'}M\rangle_{\mathcal{A}} d\mu(\omega) \leq \frac{\alpha\beta}{3}\|M\|^2_{\mathcal{A}}.$$
Which show that the family $(\Lambda_\omega)_{\omega\in\Omega}$ is a continuous $(C, C^{'})$-controlled Bessel sequence for $\mathcal{H}$ with $\frac{\alpha\beta}{3}$ as bound.
But if $b = c = 0$, it's impossible to found a positive scalar A such that
$$A\|M\|^2_{\mathcal{A}} \leq \int_{\Omega}\langle \Lambda_w CM, \Lambda_w C^{'}M\rangle_{\mathcal{A}} d\mu(\omega) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
where
$$M = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \qquad and \quad a, b > 0.$$
So, $(\Lambda_\omega)_{\omega\in\Omega}$ is not a continuous $(C, C^{'})$-controlled frame for $\mathcal{H}$.
But, if we consider the operator
$$\begin{array}{cccc} K: & \mathcal{H} & \longrightarrow & \mathcal{H} \\ & \begin{pmatrix} a & b & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix} & \longrightarrow & \begin{pmatrix} 0 & b & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}. \end{array}$$
Wich's linear, bounded and selfadjoint, we found
$$\langle K^*M, K^*M\rangle = \begin{pmatrix} |b|^2 & b\bar{c} \\ c\bar{b} & |c|^2 \end{pmatrix}.$$
Then $(\Lambda_\omega)_{\omega\in\Omega}$ is a continuous $(C, C^{'})$-controlled K-g-frame for $\mathcal{H}$.

Remark 2.4. Every continuous $(C, C')-$controlled g-frame for $\mathcal{H}$ is a continuous $(C, C')-$controlled K-g-frame for $\mathcal{H}$. Indeed, if $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C, C^{'})$-controlled g-frame for Hilbert C^*-module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$, then there exist a constants $A, B > 0$ such that ,
$$A\langle f, f\rangle_{\mathcal{A}} \leq \int_{\Omega}\langle \Lambda_\omega Cf, \Lambda_\omega C'f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathcal{A}}, f \in \mathcal{H}.$$
But,
$$\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \|K\|^2\langle f, f\rangle_{\mathcal{A}}, f \in \mathcal{H}.$$
So,
$$A\|K\|^{-2}\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \int_{\Omega}\langle \Lambda_\omega Cf, \Lambda_\omega C'f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathcal{A}}, f \in \mathcal{H}.$$
Hence, $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C, C^{'})$-controlled K-g-frame for Hilbert C^*-module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.

Let $\{\Lambda_\omega\}_{i\in\Omega}$ be a continuous $(C, C')-$controlled Bessel K-g-frame for Hilbert C^*-module $\mathcal{H}$ over $\mathcal{A}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$ with bounds A and B.

We define the operaror $T_{(C,C^{'})}$ by:
$$T_{(C,C^{'})} : l^2(\Omega, \{\mathcal{K}_w\}_{w\in\Omega}) \to \mathcal{H},$$

such that:

$$T_{(C,C')}(\{y_w\}_{w\in\Omega}) = \int_\Omega (CC')^{\frac{1}{2}}\Lambda_\omega^* y_\omega d\mu(w), \qquad \{y_w\}_{w\in\Omega} \in l^2(\Omega, \{\mathcal{K}_w\}_{w\in\Omega}).$$

The bounded linear operator $T_{(C,C')}$ is called the (C, C') synthesis operator of Λ.
The operator:

$$T^*_{(C,C')} : \mathcal{H} \to l^2(\Omega, \{\mathcal{K}_w\}_{w\in\Omega}),$$

is given by:

$$T^*_{(C,C')}(x) = \{\Lambda_\omega (C'C)^{\frac{1}{2}} x\}_{\omega\in\Omega}, \qquad x \in \mathcal{H}, \tag{2.3}$$

is called the (C, C') analysis operator for Λ.
Indeed, we have for all $x \in \mathcal{H}$ and $\{y_w\}_{w\in\Omega} \in l^2(\Omega, \{\mathcal{K}_w\}_{w\in\Omega})$

$$\begin{aligned}
\langle T_{(C,C')}(\{y_w\}_{w\in\Omega}), x\rangle_\mathcal{A} &= \langle \int_\Omega (CC')^{\frac{1}{2}}\Lambda_\omega^* y_\omega d\mu(w), x\rangle_\mathcal{A} \\
&= \int_\Omega \langle (CC')^{\frac{1}{2}}\Lambda_\omega^* y_\omega, x\rangle_\mathcal{A} d\mu(w) \\
&= \int_\Omega \langle y_\omega, \Lambda_\omega (CC')^{\frac{1}{2}} x\rangle_\mathcal{A} d\mu(w) \\
&= \langle \{y_w\}_{w\in\Omega}, \{\Lambda_\omega (C'C)^{\frac{1}{2}} x\}_{\omega\in\Omega}\rangle_{l^2(\Omega,\{\mathcal{K}_w\}_{w\in\Omega})} \\
&= \langle \{y_w\}_{w\in\Omega}, T^*_{(C,C')}(x)\rangle_{l^2(\Omega,\{\mathcal{K}_w\}_{w\in\Omega})}.
\end{aligned}$$

Which shows that $T^*_{(C,C')}$ is the adjoint of $T_{(C,C')}$. If C and C' commute between them, and commute with the operators $\Lambda_\omega^*\Lambda_\omega$ for each $\omega \in \Omega$. We define the frame operator by:

$$\begin{aligned}
S_{(C,C')} :&\mathcal{H} \longrightarrow \mathcal{H} \\
&x \longrightarrow S_{(C,C')}x = T_{(C,C')}T^*_{(C,C')}x = \int_\Omega C'\Lambda_w^*\Lambda_w C x d\mu(w).
\end{aligned}$$

As consequence on has the following proposition.

Proposition 2.5. *The operator $S_{(C,C')}$ is positive, sefladjoint, and bounded.*

Proposition 2.6. *Let $K \in End^*_\mathcal{A}(\mathcal{H})$ and $C, C' \in GL^+(\mathcal{H})$. Suppose that C and C' commutes with each other and commute with the operators $\Lambda_\omega^*\Lambda_\omega$ for each $\omega \in \Omega$. A family $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous (C, C')-controlled Bessel K-g-frames for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$ with bounds B if and only if the operator $T_{(C,C')}$ is well defined and bounded with $\|T_{(C,C')}\| \leq \sqrt{B}$.*

Proof. $(1) \Longrightarrow (2)$
Let $\{\Lambda_w, w \in \Omega\}$ be a (C, C')-controlled continuous K-g-Bessel family for $\mathcal{H}$ with respect $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ with bound B.
Then we have,

$$\| \int_\Omega \langle \Lambda_w Cx, \Lambda_w C'x\rangle_\mathcal{A} d\mu(w)\| \leq B\|x\|^2, \qquad x \in \mathcal{H}. \tag{2.4}$$

For all $\{y_w\}_{w\in\Omega} \in l^2(\Omega, \{\mathcal{K}_w\}_{w\in\Omega})$, we have,

$$\|T_{CC'}(\{y_w\}_{w\in\Omega})\|^2 = \sup_{x\in\mathcal{H},\|x\|=1} \|\langle T_{CC'}(\{y_w\}_{w\in\Omega}), x\rangle_\mathcal{A}\|^2.$$

Hence,

$$\begin{aligned}
\|T_{CC'}(\{y_w\}_{w\in\Omega})\|^2 &= \sup_{x\in\mathcal{H},\|x\|=1}\|\langle\int_\Omega (CC^{'})^{\frac{1}{2}}\Lambda_\omega^* y_\omega d\mu(w),x\rangle_\mathcal{A}\|^2\\
&= \sup_{x\in\mathcal{H},\|x\|=1}\|\int_\Omega \langle (CC^{'})^{\frac{1}{2}}\Lambda_\omega^* y_\omega,x\rangle_\mathcal{A} d\mu(w)\|^2\\
&= \sup_{x\in U,\|x\|=1}\|\int_\Omega \langle y_\omega,\Lambda_\omega (CC^{'})^{\frac{1}{2}}x\rangle_\mathcal{A} d\mu(w)\|^2\\
&\leq \sup_{x\in U,\|x\|=1}\|\int_\Omega \langle y_\omega,y_\omega\rangle_\mathcal{A} d\mu(w)\|\|\int_\Omega \langle \Lambda_\omega (CC^{'})^{\frac{1}{2}}x,\Lambda_\omega (CC^{'})^{\frac{1}{2}}x\rangle_\mathcal{A} d\mu(w)\|\\
&= \sup_{x\in U,\|x\|=1}\|\int_\Omega \langle y_\omega,y_\omega\rangle_\mathcal{A} d\mu(w)\|\|\int_\Omega \langle \Lambda_\omega Cx,\Lambda_\omega C^{'}x\rangle_\mathcal{A} d\mu(w)\|\\
&\leq \sup_{x\in\mathcal{H},\|x\|=1}\|\int_\Omega \langle y_\omega,y_\omega\rangle_\mathcal{A} d\mu(w)\|B\|x\|^2 = B\|\{y_\omega\}_{\omega\in\Omega}\|^2.
\end{aligned}$$

Then we have

$$\|T_{CC'}(\{y_w\}_{w\in\Omega})\|^2 \leq B\|\{y_\omega\}_{\omega\in\Omega}\|^2 \Longrightarrow \|T_{CC'}\| \leq \sqrt{B}.$$

We conclude that the operator $T_{CC'}$ is well defined and bounded.
(2) $\Longrightarrow$ (1)
If (2) holds, then for any $x \in \mathcal{H}$, we have:

$$\begin{aligned}
\int_\Omega \langle \Lambda_w Cx,\Lambda_w C^{'}x\rangle_\mathcal{A} d\mu(w) &= \int_\Omega \langle C^{'}\Lambda_w^*\Lambda_w Cx,x\rangle_\mathcal{A} d\mu(w)\\
&= \int_\Omega \langle (CC^{'})^{\frac{1}{2}}\Lambda_w^*\Lambda_w (CC^{'})^{\frac{1}{2}}x,x\rangle_\mathcal{A} d\mu(w)\\
&= \int_\Omega \langle \Lambda_w (CC^{'})^{\frac{1}{2}}x,\Lambda_w (CC^{'})^{\frac{1}{2}}x\rangle_\mathcal{A} d\mu(w)\\
&= \langle \{\Lambda_w (CC^{'})^{\frac{1}{2}}x\}_{\omega\in\Omega},\{\Lambda_w (CC^{'})^{\frac{1}{2}}x\}_{\omega\in\Omega}\rangle\\
&= \langle T^*_{(C,C')}(x),T^*_{(C,C')}(x)\rangle.
\end{aligned}$$

Or,

$$\langle T^*_{(C,C')}(x),T^*_{(C,C')}(x)\rangle \leq \|T^*_{(C,C')}\|^2\langle x,x\rangle_\mathcal{A}.$$

As $\|T_{CC'}\| \leq \sqrt{B}$, we have :

$$\int_\Omega \langle \Lambda_w Cx,\Lambda_w C^{'}x\rangle_\mathcal{A} d\mu(w) \leq B\|x\|^2,$$

which end the proof. □

Lemma 2.7. *Let $\{\Lambda_\omega\}_{w\in\Omega} \subset End_\mathcal{A}^*(\mathcal{H},\mathcal{K}_w)$ be a continuous $(C,C')-$controlled Bessel K-g-frame for Hilbert C^*- module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$. Then for any $K \in End_\mathcal{A}^*(\mathcal{H})$, the family $\{\Lambda_\omega K\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled Bessel K-g-frame for Hilbert C^*-module $\mathcal{H}$.*

Proof. Assume that $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled Bessel K-g-frame for Hilbert C^*- module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w\in\Omega\}$ with bound B. Then,

$$\int_\Omega \langle \Lambda_\omega Cf, \Lambda_\omega C'f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle f,f\rangle_{\mathcal{A}}, f\in\mathcal{H}.$$

So,

$$\int_\Omega \langle \Lambda_\omega CKf, \Lambda_\omega C'Kf\rangle_{\mathcal{A}} d\mu(w) \leq B\langle Kf,Kf\rangle_{\mathcal{A}}, f\in\mathcal{H}.$$

Hence,

$$\int_\Omega \langle \Lambda_\omega KCf, \Lambda_\omega KC'f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle Kf,Kf\rangle_{\mathcal{A}} \leq \|K\|^2 B\langle f,f\rangle_{\mathcal{A}}, f\in\mathcal{H}.$$

The results holds. □

Lemma 2.8. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $C,C' \in GL^+(\mathcal{H})$. Let $\{\Lambda_\omega\}_{w\in\Omega}$ be a continuous $(C,C')-$controlled Bessel K-g-frame for Hilbert C^*- module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w\in\Omega\}$. $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled K- g-frame if and only if there exists a constant $A>0$ such that*

$$AKK^* \leq S_{(C,C')}.$$

Proof. The family $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled K- g-frame if and only if

$$A\langle K^*f,K^*f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega Cf, \Lambda_\omega C'f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle f,f\rangle_{\mathcal{A}}, f\in\mathcal{H}. \tag{2.5}$$

If and only if,

$$\langle AKK^*f,f\rangle_{\mathcal{A}} \leq \langle S_{(C,C')}f,f\rangle_{\mathcal{A}} \leq \langle Bf,f\rangle_{\mathcal{A}}, f\in\mathcal{H}. \tag{2.6}$$

If

$$A\langle K^*f,K^*f\rangle_{\mathcal{A}} \leq \langle Sf,f\rangle_{\mathcal{A}},$$

and the family $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled Bessel K-g-frame sequence then:

$$\langle Sf,f\rangle_{\mathcal{A}} \leq B\langle f,f\rangle_{\mathcal{A}}, f\in\mathcal{H}.$$

Wich completes the proof.

□

Theorem 2.9. *Let $K\in End^*_{\mathcal{A}}(\mathcal{H})$ and $C,C'\in GL^+(\mathcal{H})$. Suppose that K^* commute with C and C'. If $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled g-frame for Hilbert C^*- module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w\in\Omega\}$, then $\{\Lambda_\omega K^*\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled K- g-frame for Hilbert C^*- module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w\in\Omega\}$.*

Proof. Let $\{\Lambda_\omega\}_{w\in\Omega}$ be a continuous $(C,C')-$controlled g-frame for Hilbert C^*- module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w\in\Omega\}$, then,

$$A\langle f,f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega Cf, \Lambda_\omega C'f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle f,f\rangle_{\mathcal{A}}, f\in\mathcal{H}. \tag{2.7}$$

Hence,

$$A\langle K^*f,K^*f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega CK^*f, \Lambda_\omega C'K^*f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle K^*f,K^*f\rangle_{\mathcal{A}}, f\in\mathcal{H}.$$

Therefore,

$$A\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega K^* Cf, \Lambda_\omega K^* C'f\rangle_{\mathcal{A}} d\mu(w) \leq B\|K^*\|^2\langle f, f\rangle_{\mathcal{A}}, f \in \mathcal{H}.$$

We This conclude that $\{\Lambda_\omega K^*\}_{w\in\Omega}$ is a continuous $(C, C')-$controlled K- g-frame for Hilbert C^*- module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$. □

Lemma 2.10. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $C, C' \in GL^+(\mathcal{H})$. Suppose that C and C' commute with each other and commute with S. Then $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C, C')-$controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$ if and only if $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C'C, I_{\mathcal{H}})-$controlled K-g-frame for Hilbert C^*- module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.*

Proof. For all $f \in \mathcal{H}$ we have,

$$\begin{aligned}\langle (C')^{-1}S_{(C,C')}C^{-1}f, f\rangle_{\mathcal{A}} &= \int_\Omega \langle C'\Lambda^*_\omega\Lambda_\omega CC^{-1}f, (C')^{-1}f\rangle_{\mathcal{A}} d\mu(w) \\ &= \int_\Omega \langle \Lambda^*_\omega\Lambda_\omega f, f\rangle_{\mathcal{A}} d\mu(w) \\ &= \langle Sf, f\rangle_{\mathcal{A}},\end{aligned}$$

where

$$Sf = \int_\Omega \Lambda^*_\omega\Lambda_\omega f d\mu(w).$$

Hence,

$$S = (C')^{-1}S_{(C,C')}C^{-1}$$

For any $f \in \mathcal{H}$, we have,

$$\begin{aligned}\int_\Omega \langle \Lambda_\omega Cf, \Lambda_\omega C'f\rangle_{\mathcal{A}} d\mu(w) &= \int_\Omega \langle C'\Lambda^*_\omega\Lambda_\omega Cf, f\rangle_{\mathcal{A}} d\mu(w) \\ &= \langle S_{(C,C')}f, f\rangle_{\mathcal{A}} \\ &= \langle C'SCf, f\rangle_{\mathcal{A}} \\ &= \langle CSC'f, f\rangle_{\mathcal{A}} \\ &= \langle SC'Cf, f\rangle_{\mathcal{A}} \\ &= \int_\Omega \langle \Lambda^*_\omega\Lambda_\omega C'Cf, f\rangle_{\mathcal{A}} d\mu(w) \\ &= \int_\Omega \langle \Lambda_\omega C'Cf, \Lambda_\omega f\rangle_{\mathcal{A}} d\mu(w)\end{aligned}$$

Hence, $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(CC', I_{\mathcal{H}})-$controlled K-g-frame for $\mathcal{H}$ with bounds A and B respect to $\{\mathcal{K}_w : w \in \Omega\}$ if and only if,

$$A\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega C'Cf, \Lambda_\omega f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathcal{A}}, f \in \mathcal{H}.$$

The results holds. □

Lemma 2.11. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $C, C' \in GL^+(\mathcal{H})$. Then $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C, C')-$controlled K-g-frame for Hilbert C^*-module $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$ if and* *only if $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $((C'C)^{\frac{1}{2}}, ((C'C)^{\frac{1}{2}})-$controlled K-g-frame for Hilbert $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.*

Proof. The proof is similar as proof of lemma 2.10.

□

Proposition 2.12. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $C, C' \in GL^+(\mathcal{H})$. Let $\{\Lambda_\omega\}_{w\in\Omega}$ be a continuous $(C, C')-$controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$. Suppose that $R(K)$ is closed. If $T \in End^*_{\mathcal{A}}(\mathcal{H})$ with $R(T) \subset R(K)$, then $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C, C')-$controlled T-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.*

Proof. Let $\{\Lambda_\omega\}_{w\in\Omega}$ be a continuous $(C, C')-$controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$. Then there exists $A, B > 0$ such that,

$$A\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega Cf, \Lambda_\omega C'f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathcal{A}}.$$

From lemma 1.3 and $R(T) \subset R(K)$, there exists some $m > 0$ such that

$$TT^* \leq mKK^*.$$

Hence,

$$\frac{A}{m}\langle T^*f, T^*f\rangle_{\mathcal{A}} \leq A\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega Cf, \Lambda_\omega C'f\rangle_{\mathcal{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathcal{A}}.$$

So, $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C, C')-$controlled T-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.

□

Theorem 2.13. *Let $K_1, K_2 \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $R(K_1) \perp R(K_2)$. If $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C, C')-$controlled K_1-g-frame for $\mathcal{H}$ as well a K_2-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$ and α , β are scalers. Then $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C, C')-$controlled $(\alpha K_1 + \beta K_2)$-g-frame and a continuous $(C, C')-$controlled (K_1K_2)-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.*

Proof. Let $\{\Lambda_\omega\}_{w\in\Omega} \subset End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_w)$ be a continuous $(C, C')-$controlled K_1-g-frame for $\mathcal{H}$ as well a K_2-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w : w \in \Omega\}$.
Then there exist positive constants A_1, A_2, B_1, B_2 such that,

$$A_1\langle K_1^*f, K_1^*f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega Cf, \Lambda_\omega C'f\rangle_{\mathcal{A}} d\mu(w) \leq B_1\langle f, f\rangle_{\mathcal{A}}.$$

$$A_2\langle K_2^*f, K_2^*f\rangle_{\mathcal{A}} \leq \int_\Omega \langle \Lambda_\omega Cf, \Lambda_\omega C'f\rangle_{\mathcal{A}} d\mu(w) \leq B_2\langle f, f\rangle_{\mathcal{A}}.$$

For any $f \in \mathcal{H}$, we have,

$$\langle(\alpha K_1 + \beta K_2)^*f, (\alpha K_1 + \beta K_2)^*f\rangle_{\mathcal{A}} = \langle\overline{\alpha}K_1^*f + \overline{\beta}K_2^*f, \overline{\alpha}K_1^*f + \overline{\beta}K_2^*f\rangle_{\mathcal{A}}$$

$$= |\alpha|^2\langle K_1^*f, K_1^*f\rangle_{\mathcal{A}} + \overline{\alpha}\beta\langle K_1^*f, K_2^*f\rangle + \alpha\overline{\beta}\langle K_2^*f, K_1^*f\rangle + |\beta|^2\langle K_2^*f, K_1^*f\rangle$$

.

Since $R(K_1) \perp R(K_2)$,then,

$$\langle(\alpha K_1+\beta K_2)^*f,(\alpha K_1+\beta K_2)^*f\rangle_{\mathcal{A}}=|\alpha|^2\langle K_1^*f,K_1^*f\rangle+|\beta|^2\langle K_2^*f,K_1^*f\rangle_{\mathcal{A}}.$$

Therefore, for each $f\in\mathcal{H}$, we have,

$$\frac{A_1A_2}{2(|\alpha|^2A_2+|\beta|^2A_1)}\langle(\alpha K_1+\beta K_2)^*f,(\alpha K_1+\beta K_2)^*f\rangle_{\mathcal{A}}$$

$$=\frac{A_1A_2|\alpha|^2}{2(|\alpha|^2A_2+|\beta|^2A_1)}\langle K_1^*f,K_1^*f\rangle_{\mathcal{A}}+\frac{A_1A_2|\beta|^2}{2(|\alpha|^2A_2+|\beta|^2A_1)}\langle K_2^*f,K_2^*f\rangle_{\mathcal{A}}$$

$$\leq\frac{1}{2}\int_\Omega\langle\Lambda_\omega Cf,\Lambda_\omega C'f\rangle_{\mathcal{A}}d\mu(w)+\frac{1}{2}\int_\Omega\langle\Lambda_\omega Cf,\Lambda_\omega C'f\rangle_{\mathcal{A}}d\mu(w)\leq\frac{B_1+B_2}{2}\langle f,f\rangle_{\mathcal{A}}.$$

Therefore $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled $(\alpha K_1+\beta K_2)$-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w:w\in\Omega\}$.
Also for every $f\in\mathcal{H}$ we have,

$$\begin{aligned}\langle(K_1K_2)^*f,(K_1K_2)^*f\rangle_{\mathcal{A}}&=\langle K_2^*K_1^*f,K_2^*K_1^*f\rangle_{\mathcal{A}}\\&\leq\|K_2^*\|^2\langle K_1^*f,K_1^*f\rangle_{\mathcal{A}}.\end{aligned}$$

Since $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled K_1-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w:w\in\Omega\}$, we have for every $f\in\mathcal{H}$,

$$A_1\|K_2^*\|^{-2}\langle(K_1K_2)^*f,(K_1K_2)^*f\rangle_{\mathcal{A}}\leq\int_\Omega\langle\Lambda_\omega Cf,\Lambda_\omega C'f\rangle_{\mathcal{A}}d\mu(w)\leq B_1\langle f,f\rangle_{\mathcal{A}}.$$

So, $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled (K_1K_2)-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w:w\in\Omega\}$. ☐

Corollary 2.14. *Let $K\in End_{\mathcal{A}}^*(\mathcal{H})$. If $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w:w\in\Omega\}$, then for any operator $\ominus$ in the subalgebra generated by K, the family $\{\Lambda_\omega\}_{w\in\Omega}$ is a continuous $(C,C')-$controlled $\ominus$-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_w:w\in\Omega\}$.*

Declarations

Competing interest

The authors declare that they have no competing interests.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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