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Several Properties of the Matrix Operators on the Weighted Difference Sequence Space Derived by Real Numbers r and s

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ABSTRACT. We can describe the norm for an operator given as $T : X \to Y$ as follows: It is the most suitable value of U satisfying the following inequality

$$\|Tx\|_Y \leq U\|x\|_X$$

and also for the lower bound of T we can say that the value of L conforms to the following inequality

$$\|Tx\|_Y \geq L\|x\|_X,$$

where $\|.\|_X$ and $\|.\|_Y$ stand for the norms corresponding to the X and Y spaces, respectively. The most important characteristics of the present article lies in its computing the norms and lower bounds of those matrix operators used as weighted sequence space $\ell_p(w)$ upon a new one. This new sequence space is the generalized weighted sequence one. For this aim, the difference matrix $B(r,s)$ and also the space consisting of those sequence whose $B(r,s)$ transforms lie inside $\ell_p(\tilde{w})$, in which r, s are taken from $\mathbb{R}$.

1. Introduction

We now remind some basic definitions and conclusions, which we will mainly use in the following sections. First of all, we will present the concept of the sequence, the details of which are well known acquaintance in elementary analysis. Even though there are many different ways to describe the sequence, which all mean the same thing, we chose to give following definition here. The sentence "x is a sequence" will mean $x := \{x_n\} := \{x_0, x_1, \ldots, x_n, \ldots\}$, where each x_n is a complex number. In other words, a sequence can easily be introduced as an ordered list of complex numbers. Thus if x is a sequence, then it may be viewed as a mapping $x : \mathbb{N} := \{1, 2, \ldots\} \to \mathbb{C}$. In more general terms, any x sequence in X is a transformation $x : \mathbb{N} \to X$, where X is a non-empty set. The collection of all real or complex number sequences forms a vector space which we denote by w, under the operations of coordinate-wise addition and well-known scalar multiplication. The subspaces of ω are important in such applications because each of them is called a sequence space.

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When an infinite matrix $A = (a_{nk})$ is given having complex numbers a_{nk} as entries in which $n, k \in \mathbb{N}$, for a sequence x, it can be written as

$$(Ax)_n := \sum a_{nk}x_k; \ (n \in \mathbb{N}, x \in D_{00}(A)),$$

in which $D_{00}(A)$ describes the defined subspace of ω composed of $x \in \omega$ for which the summation do exist as a finite sum. For a simple notation, from now on, the summation having no limits ranges from 0 to ∞.

The X_A is known that matrix domain of an infinite matrix A for any subspace X of the all real-valued sequence space w is described as

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}$$

which is a sequence space. There are varied techniques for producing new sequence spaces out of old ones such as X. One of them is the using any matrix domain produced by an infinite matrix A such as X_A. For a brief explanation about topic, those sequence spaces, namely X and X_A, may overlap but in any case either of them may contain the other one. For detailed information the reader can refer to the book "Summability Theory and Its Applications" by Başar (2012) and therein.

In recent years we have seen a dramatic increase in constructing new sequence spaces using matrix domain in summability areas such as sequence spaces.

Many of papers by Kızmaz (1981), Et (1993), Kirişci & Başar (2010), Kara (2013), Candan (2014), Candan (2014), Candan & Güneş (2015) and İlkhan (2018) which have been examined so far have something in common, they involve matrix domain.

The finding of the best upper bound for some known matrix operators denoted by T from $\ell_p(w)$ onto $F_{w,p}$ has been tried. In connection with this statement, it should be noted that an upper bound found out for a matrix operator denoted by T defined from a sequence space X into another one denoted by Y can be given by the following value of U

$$\|Tx\|_Y \leq U\|x\|_X,$$

in which $\|.\|_X$ and $\|.\|_Y$ denote the widely known norms prescribed on the spaces X and Y, respectively. Here, U is not dependent on x. Among those, the best value of U can be characterized as the operator norm for T.

Also, several scholars have tried to find out the lower bounds for those matrix operators. This concept firstly is put forward by Lyons (1982) about the Cesàro matrix. But after that, the other ones, among others, Bennett (1986), Bennett (1992) and Jameson (1999), Jameson (2000) have investigated the lower bounds for some matrix operators defined on the sequence space denoted by ℓ_p and at the same time on the weighted sequence space denoted by $\ell_p(w)$ having the Lorentz sequence space. In a similar way, a lower bound of a matrix operator defined as $T : X \to Y$ is defined as the value of L which satisfies the following inequality

$$\|Tx\|_Y \geq L\|x\|_X.$$

This inequality can also be utilized for some functional analysis applications. To give an example, finding the necessary and sufficient conditions for which an operator has got its inverse and at the same time finding of the operator kernel including only the zero vector for this case. Because of these reasons, the knowledge of the lower bound for an operator is important. In recent years, Dehghan & Talebi (2017) have paid attention to the largest possible lower bound value about some of the matrices on the Fibonacci sequence spaces. Furthermore, Foroutannia & Roopaei (2019) take into consideration the problem of calculating both the norm and lower-upper bounds for some operators defined on weighted difference sequence spaces. One can refer to those papers writtin by Jameson & Lashkaripour (2002), Foroutannia & Lashkaripour (2010), Chen & Wang (2011), Lashkaripour & Fathi (2012), Laskaripour & Talebi (2012), Talebi & Dehghan (2013) and Denghan & Talebi (2014) and those therein for related problems about some classical sequence spaces.

In this article, it is assumed that $w = (w_n)$ and also $\tilde{w} = (\tilde{w}_n)$ are sequences consisting of positive real terms. In the present article, a new space called as the generalized weighted difference sequence space is introduced via the generalized difference matrix. Moreover, some characteristics of this sequence space are investigated. Among others, it has been observed that although this space is semi-normed one it is not necessarily a normed one. Let us remember that a semi-normed satisfy every axiom of a norm but the semi-norm of a vector must be zero without including the zero vector. Again, it is also a semi-inner product space for the value of $p = 2$. Furthermore, an isomorphism is obtained by utilizing this space. Next, the norm for some matrix operators defined on the generalized weighted difference sequence space. In the next step, the lower bound problem for the operators described from $\ell_p(w)$ into the generalized weighted difference sequence space.

2. The Sequence Space $\ell_p(\tilde{w}, B(r,s))$

We saw in the previous chapter that many issues lead to constructing new sequence space. Furthermore, the concepts we offered were inherently large. Let us start by defining the following matrix $\hat{B} = (\hat{b}_{nk}(r,s))$, which is similar but different to the matrix presented by Kirişci & Başar (2010) earlier

$$\hat{b}_{nk}(r,s) = \begin{cases} s, & k = n+1 \\ r, & k = n \\ 0, & 0 \le k < n \text{ or } k > n+1 \end{cases}$$

where $r, s \in \mathbb{R}$. We will see later that this matrix enables us to construct an efficient structure for solving algebraic and topological properties. By applying the matrix domain definition to this matrix, we define the new sequence space $\ell_p(\tilde{w}, B(r,s))$ of which result is in the $\ell_p(\tilde{w})$ space, as follows:

$$\left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \tilde{w}_n \left| r x_n + s x_{n+1} \right|^p < \infty \right\},$$

in which $1 \leq p < \infty$. We note here that, the space is a semi-normed space with the semi-norm defined by

$$\|x\|_{p,\tilde{w},B} = \left(\sum_{n=1}^{\infty} \tilde{w}_n \left| rx_n + sx_{n+1} \right|^p \right)^{1/p}.$$

To calculate the veracity of this claim, we now give an example. Considering the sequence $x_n = \frac{1}{r}(\frac{-s}{r})^{1-n}$, because of $rx_n + sx_{n+1} = 0$ we get $\|x\|_{p,\tilde{w},B} = 0$, after that, from the definition of the norm, it is seen that $\|.\|_{p,\tilde{w},B}$ defined on $\ell_p(\tilde{w}, B(r,s))$ is not a norm.

Before beginning the general theory, at first we will state the following fundamental theorem, showing that set just described have an important role in its algebraic structure.

Theorem 1. *The set $\ell_p(\tilde{w}, B(r,s))$ is linear space, namely, sequence space.*

Proof. We omit the proof which can be found in standard process. □

Let us continue with the following theorem regarding an algebraic property of this newly defined sequence space.

Theorem 2. *It is true that the inclusion relation $\ell_p(\tilde{w}) \subset \ell_p(\tilde{w}, B(r,s))$ is strictly valid.*

Proof. If we take an arbitrary $x \in \ell_p(\tilde{w})$, then following calculation demonstrates that inclusion is valid

$$\begin{aligned} \tilde{w}_n \left| rx_n + sx_{n+1} \right|^p \leq & \tilde{w}_n 2^{p-1} \left(|rx_n|^p + |sx_{n+1}|^p \right) \\ \leq & 2^{p-1} max(|r|^p, |s|^p) \\ & \quad \tilde{w}_n \left(|x_n|^p + |x_{n+1}|^p \right) \end{aligned}$$

by summing of n from 1 to ∞, in which $1 \leq p < \infty$.

To illustrate that the inclusion relation is strictly valid. When the sequence $\tilde{w}$ is taken $(1,1,1,...)$, let us consider again the sequence $(x_n) = \left(\frac{1}{r}(\frac{-s}{r})^{1-n}\right) \in \ell_p(\tilde{w}, B(r,s))$. It is easy to deduce from that $(x_n) \notin \ell_p(\tilde{w})$. □

Theorem 3. *When $H = \{x = (x_n) \in \ell_p(\tilde{w}, B(r,s)) : rx_n + sx_{n+1} = 0 \text{ for all } n \in \mathbb{N}\}$, the quotient space $\ell_p(\tilde{w}, B(r,s))/H$ is linearly isomorphic to the space $\ell_p(\tilde{w})$.*

Proof. The basic approach to the proof of this theorem is to define a new T transformation from the space $\ell_p(\tilde{w}, B(r,s))$ to $\ell_p(\tilde{w})$ that utilize the definition of the fundamental matrix transformation, for all $x \in \ell_p(\tilde{w}, B(r,s))$ clearly $Tx = (rx_n + sx_{n+1})$. Since it is quite obvious to show that T is linear, we will first be concerned here to show that T is surjective. One of the ways of doing so for any $y = (y_k) \in \ell_p(\tilde{w})$ is to write $x_n = \frac{1}{r}\sum_{k=n}^{\infty}\left(\frac{-s}{r}\right)^{k-n} y_k$ for all $n \in \mathbb{N}$ in the norm of $\ell_p(\tilde{w}, B(r,s))$. In this case, we obtain the following equations by simple calculations

$$\begin{aligned}
&\|x\|_{p,\tilde{w},B}^{p} \\
=&\sum_{n=1}^{\infty}\tilde{w}_n\left|r\sum_{k=n}^{\infty}\frac{1}{r}\left(\frac{-s}{r}\right)^{k-n}y_k\right. \\
&\qquad\left.+s\sum_{k=n+1}^{\infty}\frac{1}{r}\left(\frac{-s}{r}\right)^{k-n-1}y_k\right|^p \\
=&\sum_{n=1}^{\infty}\tilde{w}_n\left|y_n+\left[\sum_{k=n+1}^{\infty}\left(\frac{-s}{r}\right)^{k-n}y_k\right.\right. \\
&\qquad\left.\left.-\sum_{k=n+1}^{\infty}\left(\frac{-s}{r}\right)^{k-n}y_k\right]\right|^p \\
=&\sum_{n=1}^{\infty}\tilde{w}_n\left|y_n\right|^p \\
=&\|y\|_{p,\tilde{w}}^{p} \\
<&\infty
\end{aligned}$$

which implies that $x=(x_n)\in\ell_p(\tilde{w},B(r,s))$. Going back to the T transform described above, it is very straightforward to say that $Tx=y$. Because of the fact that image of the space $\ell_p(\tilde{w},B(r,s))$ under the transformation T is $\ell_p(\tilde{w})$ and also $\ker T=H$, we have that $\ell_p(\tilde{w},B(r,s))/H$ is linearly isomorphic to the space $\ell_p(\tilde{w})$ when considering the first isomorphism theorem. □

Let us give an example to show that the transformation T defined above is not injective. Indeed, for $(x_n)=(\frac{1}{r}(\frac{-s}{r})^{1-n})$ we obtain $Tx=0$; in the other word, $\ker T\neq\{0\}$.

Theorem 4. *When p is not equal to* 2 *and at the same time the space* $\ell_p(\tilde{w},B(r,s))$ *is not given as a semi-inner product space, then it is concluded that the space* $\ell_2(\tilde{w},B(r,s))$ *is defined as a semi-inner product space.*

Proof. First of all, we will answer the question that the semi-norm $\|.\|_{2,\tilde{w},B}$ using a semi-inner product can be induced. It is convenient to introduce at this stage the notation $\lambda_k=\tilde{w}_k^{1/2}(rx_k+sx_{k+1})$ for all $k\in\mathbb{N}$ and $\langle\lambda,\lambda\rangle_2=\sum_{k=1}^{\infty}|\lambda_k|^2$. In fact taken arbitrary, $x\in\ell_2(\tilde{w},B(r,s))$, we have

$$\|x\|_{2,\tilde{w},B}=\sqrt{\langle\lambda,\lambda\rangle_2}.$$

Also, it is easy to check from the following equations that the semi-norm $\|.\|_{p,\tilde{w},B}$ cannot be derived considering a semi-inner product just described

$$\|x+y\|_{p,\tilde{w},B}^2+\|x-y\|_{p,\tilde{w},B}^2=4(\tilde{w}_1^{2/p}+\tilde{w}_2^{2/p})$$

$$2(\|x\|_{p,\tilde{w},B}^2+\|y\|_{p,\tilde{w},B}^2)=4\left(\tilde{w}_1+\frac{\tilde{w}_2}{2^p}\right)^{2/p}$$

in which $x = \left(\frac{2r+s}{2r^2}, -\frac{1}{2r}, 0, 0, ...\right)$, $y = \left(\frac{2r-s}{2r^2}, \frac{1}{2r}, 0, 0, ...\right)$ and

$$4(\tilde{w}_1^{2/p} + \tilde{w}_2^{2/p}) \neq 4\left(\tilde{w}_1 + \frac{\tilde{w}_2}{2^p}\right)^{2/p}$$

for $p \neq 2$. □

3. The Norm of Matrix Operators from $\ell_1(w)$ to $\ell_1(\tilde{w}, B(r,s))$

After defining a function from the space $\ell_1(w)$ to the space $\ell_1(\tilde{w}, B(r,s))$, in this chapter we will calculate that it is a norm. Before proceeding to develop general theory, let us warm up with a very simple definition.

The matrix $\Lambda = (\lambda_{nk})$ is known as quasi-summable when Λ is the upper triangular matrix, namely, $\lambda_{nk} = 0$ for $n > k$. As can be clearly seen the matrix satisfies $\sum_{n=1}^{k} \lambda_{nk} = 1$ for all $k \in \mathbb{N}$.

Theorem 5. *The matrix $T = (t_{nk})$ is a bounded matrix operator from the space $\ell_1(w)$ to the space $\ell_1(\tilde{w}, B(r,s))$ if $M = \sup_{k\in\mathbb{N}} \frac{s_k}{w_k} < \infty$, in which $\lambda_k = \sum_{n=1}^{\infty} \tilde{w}_n |rt_{nk} + st_{n+1,k}|$. In that case, the norm of operator is obtained as $\|T\|_{1,w,\tilde{w},B} = M$.*

For all $n \in \mathbb{N}$, taking both $w_n = 1$ and $\tilde{w}_n = 1$ specially, the transformation T is a bounded operator from the space ℓ_1 to the space $\ell_1(B(r,s))$ and also $\|T\|_{1,B} = \sup_{k\in\mathbb{N}} s_k$.

Proof. We consider a sequence $x = (x_n)$ in $\ell_1(w)$, therefore

$$\begin{aligned}
\|Tx\|_{1,\tilde{w},B} &= \sum_{n=1}^{\infty} \tilde{w}_n \left|\sum_{k=1}^{\infty} (rt_{nk} + st_{n+1,k})\, x_k\right| \\
&\leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \tilde{w}_n \left|rt_{nk} + st_{n+1,k}\right| |x_k| \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \tilde{w}_n \left|rt_{nk} + st_{n+1,k}\right| |x_k| \\
&= \sum_{k=1}^{\infty} \lambda_k |x_k| \\
&\leq M \sum_{k=1}^{\infty} w_k |x_k| \\
&= M\|x\|_{1,w}.
\end{aligned}$$

What is seen from these equations is that $\|T\|_{1,w,\tilde{w},B} \leq M$ since $\frac{\|Tx\|_{1,\tilde{w},B}}{\|x\|_{1,w}} \leq M$. We proceed by introducing the sequence $e^i = (0, 0, ..., 0, \overset{i.}{1}, 0, ...)$ for each $i \in \mathbb{N}$ for computing the converse inequality, and then obtain $\|e^i\|_{1,w} = w_i$ and also $\|Te^i\|_{1,\tilde{w},B} = \lambda_i$. Because of these, it is easy to see that $\|T\|_{1,w,\tilde{w},B} \geq M$ first, and then $\|T\|_{1,w,\tilde{w},B} = M$. □

Theorem 6. *Let us assume that $S = (s_{nk})$ is the upper triangular matrix having the non-negative entries and also assume that (w_n) is an increasing given sequence. When the inequality $s_{nk} \geq s_{n+1,k}$ is valid for each values of $n \in \mathbb{N}$, constant $k \in \mathbb{N}$, at the same time $s = -r < 0$ and $M' = \sup_{k\in\mathbb{N}} s_{1k} < \infty$, then S is defined as a bounded operator described* *from $\ell_1(w)$ to $\ell_1(w, B(r,s))$. At the same time, the norm of this given operator satisfies the inequality given in the form $\|S\|_{1,w,B} \leq rM'$. When the specific condition of S is being quasi summable matrix is taken into consideration, thus the condition $\|S\|_{1,w,B} = r$ is obtained.*

Proof. Due to the hypothesis, we need to say that the matrix $S = (s_{nk})$ that satisfies the $s_{nk} \geq s_{n+1,k}$ (for all $n, k = 1, 2, ...$) condition is upper triangular and also the sequence (w_n) is increasing. With simple calculations and taking into consideration $s = -r < 0$, the following is derived

$$\begin{aligned}
\lambda_k &= \sum_{n=1}^{\infty} w_n \left| r s_{nk} - r s_{n+1,k} \right| \\
&= \sum_{n=1}^{k-1} w_n \left| r s_{nk} - r s_{n+1,k} \right| + w_k |r| s_{kk} \\
&\leq w_k \left[\sum_{n=1}^{k-1} (r s_{nk} - r s_{n+1,k}) + r s_{kk} \right] \\
&= w_k \left[(r s_{1k} - r s_{2k}) + ... + r s_{kk} \right] \\
&= w_k \left[r s_{1k} + (r - r)\, s_{2k} + ... + (r - r)\, s_{kk} \right] \\
&= r w_k s_{1k}.
\end{aligned}$$

Clearly, $\|S\|_{1,w,B} = \sup_{k\in\mathbb{N}} \frac{s_k}{w_k} \leq rM' = r \sup_{k\in\mathbb{N}} s_{1k}$ from Theorem 5.

Let us assume that S is a quasi summable matrix, therefore $M' \leq 1$ and hence $\|S\|_{1,w,B} \leq r$. To get the inverse inequality, let us take into account the sequence $e^1 = (1, 0, 0, ...)$. From this it follows that $\|e^1\|_{1,w} = w_1$ and $\|Se^1\|_{1,w,B} = w_1$, namely $\|S\|_{1,w,B} \geq r$. As a result, $\|S\|_{1,w,B} = r$ is obtained. □

Let us state that, in Theorem 6, if there is no constriction on r and s, then since

$$\begin{aligned}
\lambda_k &= \sum_{n=1}^{\infty} w_n \left| r s_{nk} + s s_{n+1,k} \right| \\
&\leq (|r| + |s|) w_k \sup_{n\in\mathbb{N}} \sum_{n=1}^{k} s_{nk}
\end{aligned}$$

the inequality is valid, the following is obtained $\|S\|_{1,w,B} \leq (|r| + |s|)M'$, where $M' = \sup_{k\in\mathbb{N}} \sum_{n=1}^{k} s_{nk}$.

In light of the above mentioned theorems, we are here concerned with calculating the norm of some specific quasi summable matrices. Initially, we consider the transpose of the well-known Riesz matrix $\tilde{R} = (\tilde{r}_{nk})$ described as follows:

$$\tilde{r}_{nk} = \begin{cases} \frac{q_n}{Q_k}, & n \leq k \\ 0, & n > k, \end{cases} \tag{1}$$

in which (q_n) is a non-negative sequence with $q_1 > 0$ and $Q_k = q_1 + ... + q_k$ for all $k \in \mathbb{N}$. If we take $q_n = 1$ for all $n \in \mathbb{N}$, we derive the transpose of the Cesàro matrix of order one which is also known as Copson matrix (Jameson 2000). We indicate this specific matrix by $\tilde{C} = (\tilde{c}_{nk})$, in which

$$\tilde{c}_{nk} = \begin{cases} \frac{1}{k}, & n \le k \\ 0, & n > k. \end{cases}$$

Corollary 7. *When (q_n) is a decreasing sequence and (w_n) is an increasing sequence, in that case $\tilde{R}$ is a bounded operator from the space $\ell_1(w)$ into the space $\ell_1(w, B(r,s))$ and, also $\|\tilde{R}\|_{1,w,\tilde{B}} = r$ for $s = -r < 0$.*

Proof. First of all, since (q_n) is a decreasing sequence from the hypothesis the following inequality $\tilde{r}_{nk} = \frac{q_n}{Q_k} \ge \frac{q_{n+1}}{Q_k} = \tilde{r}_{n+1,k}$ holds for all $n \in \mathbb{N}$, each fixed $k \in \mathbb{N}$. For $\tilde{R}$ is a non-negative upper triangular matrix and (w_n) is an increasing sequence, it follows from Theorem 6 that $\tilde{R}$ is a bounded operator from $\ell_1(w)$ into $\ell_1(w, B(r,s))$. Also due to the fact that $\sum_{n=1}^{k} \tilde{r}_{nk} = 1$ for every $k \in \mathbb{N}$, $\tilde{R}$ is a quasi summable matrix. If $s = -r < 0$ then it is clear that $\|\tilde{R}\|_{1,w,B} = r$ from Theorem 6. □

Corollary 8. *If $\sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k} < \infty$, then the matrix $\tilde{C}$ defined just above is a bounded operator from the space $\ell_1(w)$ into $\ell_1(\tilde{w}, B(r,s))$ and $\|\tilde{C}\|_{1,w,\tilde{w},B} \le (|r| + |s|) \sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k}$.*

Proof. We get the following inequality

$$\begin{aligned} \lambda_k &= \sum_{n=1}^{\infty} \tilde{w}_n \left| r\tilde{c}_{nk} + s\tilde{c}_{n+1,k} \right| \\ &\le \frac{1}{k} \left[\sum_{n=1}^{k-1} \tilde{w}_n \left(|r| + |s| \right) + \tilde{w}_k |r| \right] \\ &= \frac{|r|}{k} \sum_{n=1}^{k} \tilde{w}_n + \frac{|s|}{k} \sum_{n=1}^{k-1} \tilde{w}_n \\ &\le \frac{|r| + |s|}{k} \sum_{n=1}^{k} \tilde{w}_n. \end{aligned}$$

for $r, s \in \mathbb{R}$, therefore we obtain that $\|\tilde{C}\|_{1,w,\tilde{w},B} \le (|r|+|s|) \sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k}$ from Theorem 5. □

When the result obtained for the special cases of r and s is compared with the result obtained without any restrictions, since the general result is greater than the particular ones, there is not any contradictory situation.

Theorem 9. *Let us suppose that $T = (t_{nk})$ is a matrix having the non-negative entries and the inequalities $t_{nk} \ge t_{n+1,k}$ hold for all $n \in \mathbb{N}$ and each fixed $k \in \mathbb{N}$ and $r \ge -s > 0$ are valid. If $\sum_{n=1}^{\infty} t_{nk} < \infty$ for each $k \in \mathbb{N}$ and also $M'' = \sup_{k\in\mathbb{N}} \sum_{n=1}^{\infty} t_{nk} < \infty$, then the matrix T is a bounded operator from the space ℓ_1 to $\ell_1(B(r,s))$ and the norm of operator is $\|T\|_{1,B} \le (r+s)M''$. When the fact that the specific condition of T is being quasi summable matrix is taken into consideration for $s = -r < 0$, then the condition $\|T\|_{1,B} = r$ is derived.*

Proof. For arbitrary $k \in \mathbb{N}$, we have

$$s_k = \sum_{n=1}^{\infty} (rt_{nk} + st_{n+1,k}) = (r+s) \sum_{n=1}^{\infty} t_{nk}.$$

If the Theorem 5 is used here, it is found that norm $\|T\|_{1,B} \leq (r+s)M''$. The rest of the proof can be done similarly to the proof of Theorem 6. □

The matrix $H = (h_{nk})$ defined as $h_{nk} = \frac{1}{n+k}$ for all $n, k \in \mathbb{N}$ is called the Hilbert matrix operator. Here, we will discover the norm of the operator just mentioned.

Now, let us give the following integral to be used in the proofs:

$$\int_0^{\infty} \frac{1}{t^{\alpha}(t+c)} dt = \frac{\pi}{c^{\alpha} \sin \alpha\pi},$$

in which $0 < \alpha < 1$.

Theorem 10. *Let $w_n = \frac{1}{n^{\alpha}}$ for all $n \in \mathbb{N}$, in which $0 < \alpha < 1$. In this case, the Hilbert matrix operator H just described is bounded from the space $\ell_1(w)$ to the space $\ell_1(w, B(r,s))$ and also the norm $\|H\|_{1,w,B} \leq \frac{\pi}{\sin \alpha\pi}(|r| + |s|)$.*

Proof. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\lambda_k &= \sum_{i=1}^{\infty} w_i \left| rh_{in} + sh_{i+1,n} \right| \\
&\leq \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} \left(\frac{|r|}{i+n} + \frac{|s|}{i+n+1} \right) \\
&\leq \int_0^{\infty} \frac{1}{t^{\alpha}} \left(\frac{|r|}{t+n} + \frac{|s|}{t+n+1} \right) dt \\
&= \frac{\pi}{\sin \alpha\pi} \left(\frac{|r|}{n^{\alpha}} + \frac{|s|}{(n+1)^{\alpha}} \right).
\end{aligned}$$

It follows that

$$\begin{aligned}
n^{\alpha} \lambda_n &\leq \frac{\pi}{\sin \alpha\pi} \left[|r| + |s| \left(\frac{n}{n+1} \right)^{\alpha} \right] \\
&\leq \frac{\pi}{\sin \alpha\pi} \left(|r| + |s| \right).
\end{aligned}$$

Considering Theorem 5, this means that $\|H\|_{1,w,\tilde{F}} \leq \frac{\pi}{\sin \alpha\pi}(|r| + |s|)$. □

4. The Norm of Matrix Operators from $\ell_p(w)$ to $\ell_p(w, B(r,s))$

In this section, we will discuss calculating the norm of some matrix operators from the space $\ell_p(w)$ to the space $\ell_p(\tilde{w}, B(r,s))$. We now present a essential lemma which is obtained by Jameson & Lashkaripour (2000), since this important result is used in the proofs.

Lemma 11. *(Jameson 2000) Let us suppose that $S = (s_{nk})$ is a matrix operator having the nonnegative entries $s_{nk} \geq 0$, also suppose that (u_n) and (t_k) are positive sequences given such that $u_n^{1/p} \sum_{k=1}^{\infty} \frac{s_{nk}}{t_k^{1/p}} \leq A$ $(A \in \mathbb{R})$ for $n \in \mathbb{N}$ and $\frac{1}{t_k^{(1-p)/p}} \sum_{n=1}^{\infty} u_n^{(1-p)/p} s_{nk} \leq B$ $(B \in \mathbb{R})$ for $k \in \mathbb{N}$, in that case, that inequality $\|S\|_p \leq \frac{B^{1/p}}{A^{(1-p)/p}}$ is valid, in which $p > 1$.*

Now, let us state and prove another necessary lemma.

Lemma 12. *Let us assume that the equality* $s_{nk} = \left(\frac{\tilde{w}_n}{w_k}\right)^{1/p}(rt_{nk} + st_{n+1,k})$ *is valid for the matrix operators* $T = (t_{nk})$ *and* $S = (s_{nk})$*. At the same time, we have* $\|T\|_{p,w,\tilde{w},B} = \|S\|_p$*, for* $p \geq 1$*. Under the conditions of this hypothesis,* T *is bounded operator from the space* $\ell_p(w)$ *to the space* $\ell_p(\tilde{w}, B(r,s))$ *iff* S *is bounded operator onto the space* ℓ_p*.*

Proof. If the x lying in the space $\ell_p(w)$ is taken as arbitrary, and the sequence $y = (y_k)$ is defined as $y_k = w_k^{1/p} x_k$ for all $k \in \mathbb{N}$ by making use of it, then we derive that equality $\|x\|_{p,w} = \|y\|_p$. Therefore, the proof should be clear with the following rudimentary calculations

$$\begin{aligned}
I &= \sup_{x\in\ell_p(w),x\neq 0} \frac{\|Tx\|^p_{p,\tilde{w},B}}{\|x\|^p_{p,w}} \\
&= \sup_{x\in\ell_p(w),x\neq 0} \frac{\sum_{n=1}^{\infty} \tilde{w}_n \left|\sum_{k=1}^{\infty} (rt_{nk} + st_{n+1,k})\, x_k\right|^p}{\sum_{k=1}^{\infty} w_k |x_k|^p} \\
&= \sup_{y\in\ell_p} \frac{\sum_{n=1}^{\infty} \left|\sum_{k=1}^{\infty} \left(\frac{\tilde{w}_n}{w_k}\right)^{1/p} (rt_{nk} + st_{n+1,k})\, y_k\right|^p}{\sum_{k=1}^{\infty} |y_k|^p} \\
&= \sup_{y\in\ell_p} \frac{\sum_{n=1}^{\infty} \left|\sum_{k=1}^{\infty} s_{nk} y_k\right|^p}{\sum_{k=1}^{\infty} |y_k|^p} \\
&= \sup_{y\in\ell_p} \frac{\|Sy\|^p_p}{\|y\|^p_p} \\
&= \|S\|^p_p
\end{aligned}$$

where $I = \|T\|^p_{p,w,\tilde{w},B}$. □

Theorem 13. *Let us assume that the matrix operator* $\tilde{R}$ *is as defined in* (1)*, and also assume that* (q_n) *is a decreasing sequence having* $q_1 = q_2 = 2$ *and* $\lim_{n\to\infty} Q_n = \infty$*. For all* $n \in \mathbb{N}$*, if the sequence* (w_n) *is taken as* $\left(\frac{2Q_{n-1}}{q_n}\right)^p$ *with* $Q_0 = 1$*, in that case,* $\tilde{R}$ *is bounded operator from the space* $\ell_p(w)$ *to the space* $\ell_p(B(r,s))$ *and* $\|\tilde{R}\|_{p,w,B} = r$ *for* $p > 1$ *and* $s = -r < 0$*.*

Proof. In Lemma 12, utilize the matrix $\tilde{R}$ in place of T. So, the matrix $S = (s_{nk})$ is described by

$$s_{nk} = \begin{cases} \frac{rq_k}{2Q_{k-1}Q_k}\left(q_n - q_{n+1}\right), & n < k \\ \frac{1}{2} r \frac{q_k^2}{Q_{k-1}Q_k}, & n = k \\ 0, & n > k \end{cases}$$

and besides that $\|\tilde{R}\|_{p,w,B} = \|S\|_p$ is obtained.

We derive

$$\sum_{k=1}^{\infty} s_{nk} = \frac{r}{2} q_n \frac{q_n}{Q_{n-1}Q_n} + \frac{r}{2}(q_n - q_{n+1}) \sum_{k=n+1}^{\infty} \frac{q_k}{Q_{k-1}Q_k}$$
$$= \frac{r}{2} q_n \left(\frac{1}{Q_{n-1}} - \frac{1}{Q_n}\right) + \frac{r}{2}(q_n - q_{n+1}) \frac{1}{Q_n}$$
$$= \frac{r}{2}\frac{q_n}{Q_{n-1}} - \frac{r}{2}\frac{q_{n+1}}{Q_n}$$
$$\leq r$$

for all $n \in \mathbb{N}$. Also, we derive

$$\sum_{n=1}^{\infty} s_{nk} = \frac{r}{2}\frac{q_k}{Q_{k-1}Q_k}\left[\sum_{n=1}^{k-1}(q_n - q_{n+1})\right] + \frac{r}{2}\frac{q_k}{Q_{k-1}Q_k} q_k$$
$$= \frac{r}{2}\frac{q_k}{Q_{k-1}Q_k}\sum_{n=1}^{k} q_n$$
$$\leq r$$

for all $k \in \mathbb{N}$. Now, In Lemma 11, if we take $u_n = t_n = 1$ for all $n \in \mathbb{N}$, we get $A \leq r$ and $B \leq r$ which require that $\|\tilde{R}\|_{p,w,B} \leq r$. Now, for $e^1 = (1, 0, 0, ...)$, we get $\|e^1\|_{p,w} = \frac{2Q_0}{q_1} = 1$ and $\|\tilde{R}e^1\|_{p,B} = \left(\left(r\frac{q_1}{Q_1}\right)^p\right)^{\frac{1}{p}} = r$ and then $\|\tilde{R}\|_{p,w,B} \geq r$. □

Theorem 14. *Let $w_n = \frac{1}{n^\alpha}$ for all $n \in \mathbb{N}$, in which $1 - p < \alpha < 1$ and $p > 1$. In that case, the Hilbert matrix operator H is a bounded operator from the space $\ell_p(w)$ to the space $\ell_p(w, B(r, s))$ also following inequality*

$$\|H\|_{p,w,B} \leq (|r| + |s|) \max\left\{\frac{\pi}{\sin\beta\pi}, \frac{\pi}{\sin\gamma\pi}\right\},$$

is valid, in which $\beta = \frac{1-\alpha}{p}$ and $\gamma = \frac{p-1+\alpha}{p}$.

Proof. Let us define the matrix $S = (s_{nk})$ as follows

$$s_{nk} = \left(\frac{k}{n}\right)^{\alpha/p}\left(\frac{r}{n+k} + \frac{s}{n+k+1}\right)$$

for all $n, k \in \mathbb{N}$. In this case, $\|H\|_{p,w,B} = \|S\|_p$ which obtained by using Lemma 12. Specifically, when we choose $u_n = t_n = n$ in Lemma 11 for all $n \in \mathbb{N}$,if we write $I_1 = u_n^{\frac{1}{p}} \sum_{k=1}^{\infty} \frac{s_{nk}}{t_k^{\frac{1}{p}}}$, $I_2 = \frac{1}{t_k^{\frac{1-p}{p}}} \sum_{n=1}^{\infty} u_n^{\frac{1-p}{p}} s_{nk}$ and $A(r, s, n, k) = \frac{r}{n+k} + \frac{s}{n+k+1}$ for ease of writing here, we find that

$$\begin{aligned}
I_1 &= n^{1/p} \sum_{k=1}^{\infty} \frac{1}{k^{1/p}} \left(\frac{k}{n}\right)^{\alpha/p} A(r,s,n,k) \\
&\leq n^{\beta} \sum_{k=1}^{\infty} \frac{1}{k^{\beta}} \left(\frac{|r|}{n+k} + \frac{|s|}{n+k+1}\right) \\
&\leq n^{\beta} \int_{t=0}^{\infty} \frac{1}{t^{\beta}} \left(\frac{|r|}{t+n} + \frac{|s|}{t+(n+1)}\right) dt \\
&= n^{\beta} \left(\frac{|r|\pi}{n^{\beta}\sin\beta\pi} + \frac{|s|\pi}{(n+1)^{\beta}\sin\beta\pi}\right) \\
&\leq \frac{\pi}{\sin\beta\pi}\left(|r|+|s|\right)
\end{aligned}$$

for all $n \in \mathbb{N}$ also

$$\begin{aligned}
I_2 &= \frac{1}{k^{(1-p)/p}} \sum_{n=1}^{\infty} n^{(1-p)/p} \left(\frac{k}{n}\right)^{\alpha/p} A(r,s,n,k) \\
&\leq k^{\gamma} \sum_{n=1}^{\infty} \frac{1}{n^{\gamma}} \left(\frac{|r|}{n+k} + \frac{|s|}{n+k+1}\right) \\
&\leq k^{\gamma} \int_{t=0}^{\infty} \frac{1}{t^{\gamma}} \left(\frac{|r|}{t+k} + \frac{|s|}{t+(k+1)}\right) dt \\
&= k^{\gamma} \left(\frac{|r|\pi}{k^{\gamma}\sin\gamma\pi} + \frac{|s|\pi}{(k+1)^{\gamma}\sin\gamma\pi}\right) \\
&\leq \frac{\pi}{\sin\gamma\pi}\left(|r|+|s|\right)
\end{aligned}$$

for all $k \in \mathbb{N}$, where $\beta = \frac{1-\alpha}{p}$ and $\gamma = \frac{p-1+\alpha}{p}$. We therefore obtain that

$$\|H\|_{p,w,B} \leq (|r|+|s|) \max\left\{\frac{\pi}{\sin\beta\pi}, \frac{\pi}{\sin\gamma\pi}\right\},$$

from Lemma 11. □

5. Lower Bounds of Matrix Operators from $\ell_p(w)$ to $\ell_p(\tilde{w}, B(r,s))$

An important problem that arises in this work is how to calculate the lower bound of an operator T from the space $\ell_p(w)$ to space $\ell_p(\tilde{w}, B(r,s))$. Therefore, what will be done here is to obtain the lower bound of the operator T for the largest L value that satisfies the following inequality

$$\|Tx\|_{p,\tilde{w},B} \geq L\|x\|_{p,w}$$

for every decreasing sequence $x = (x_k)$ with $x_k \geq 0$.

We need the following lemma to perform the calculations in the proofs in this section.

Lemma 15. *(Jameson (2000)) Let us assume that both (q_n) and (x_n) are non-negative sequences, and that (x_n) is also a decreasing sequence satisfying condition $\lim_{n\to\infty} x_n = 0$. For $Q_n = \sum_{i=1}^{n} q_i$ with $Q_0 = 1$ also $R_n = \sum_{i=1}^{n} q_i x_i$, the following statements holds, in which $p \geq 1$ and $n \in \mathbb{N}$.*

(1) $R_n^p - R_{n-1}^p \geq (Q_n^p - Q_{n-1}^p)x_n^p$.
(2) *When the series* $\sum_{i=1}^{\infty} q_i x_i$ *converges, the following inequality is satisfied.*

$$\left(\sum_{i=1}^{\infty} q_i x_i\right)^p \geq \sum_{n=1}^{\infty} Q_n^p (x_n^p - x_{n+1}^p).$$

Theorem 16. *When* $T = (t_{nk})$*is a matrix operator with* $t_{nk} \geq 0$ *from the space* $\ell_p(w)$ *into the space* $\ell_p(\tilde{w}, B(r,s))$*, in which* $p \geq 1$*, the following inequality* $t_{nk} \geq t_{n+1,k}$ *is valid for all* $n \in \mathbb{N}$*, each fixed* $k \in \mathbb{N}$ *also the series* $\sum_{n=1}^{\infty} w_n$ *diverges to infinity, in that case, for every decreasing sequence* $x = (x_k)$ *having* $x_k \geq 0$*, we have*

$$\|Tx\|_{p,\tilde{w},B} \geq L\|x\|_{p,w},$$

in which $L^p = \inf_{n\in\mathbb{N}} \frac{S_n}{W_n}$*,* $W_n = \sum_{k=1}^{n} w_k$ *and* $S_n = \sum_{i=1}^{\infty} \tilde{w}_i \left(\sum_{k=1}^{n} (rt_{ik} + st_{i+1,k})\right)^p$ *for* $r \geqslant -s > 0$.

Proof. Under the conditions of the hypothesis expressed in the theorem, we can make the proof as follows. Since $\sum_{n=1}^{\infty} w_n = \infty$, we get $\lim_{k\to\infty} x_k = 0$, and at the same time, it can be obtained that the series $\sum_{k=1}^{\infty} (rt_{nk} + st_{n+1,k})\, x_k$ is convergent for all $n \in \mathbb{N}$. On the other hands, by using Lemma 15 and also using Abel summation, we have

$$\begin{aligned}
\|Tx\|_{p,\tilde{w},B}^p &= \sum_{n=1}^{\infty} \tilde{w}_n \left(\sum_{k=1}^{\infty} (rt_{nk} + st_{n+1,k})\, x_k\right)^p \\
&\geq \sum_{n=1}^{\infty} \tilde{w}_n \sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} (rt_{nk} + st_{n+1,k})\right)^p \\
&\quad (x_i^p - x_{i+1}^p) \\
&= \sum_{i=1}^{\infty} \left[\sum_{n=1}^{\infty} \tilde{w}_n \left(\sum_{k=1}^{i} (rt_{nk} + st_{n+1,k})\right)^p\right] \\
&\quad (x_i^p - x_{i+1}^p) \\
&= \sum_{i=1}^{\infty} S_i (x_i^p - x_{i+1}^p) \\
&\geq L^p \sum_{i=1}^{\infty} W_i (x_i^p - x_{i+1}^p) \\
&= L^p \|x\|_{p,w}^p
\end{aligned}$$

which completes the proof. □

The following lemma can be verified in a similar technique with the proof of Proposition 1 in Jameson (2000).

Lemma 17. *Let us assume that* $T = (t_{nk})$ *be a non-negative matrix operator defined from the space* $\ell_p(w)$ *to the space* $\ell_p(\tilde{w}, B(r,s))$*, in which* $p \geq 1$*. If the following inequality*

$$rt_{nk} + st_{n+1,k} \geq rt_{n,k+1} + st_{n+1,k+1}$$

is valid also $t_{nk} \geq t_{n+1,k}$ for all $k \in \mathbb{N}$, each fixed $n \in \mathbb{N}$ and $r \geq -s > 0$, if the series $\sum_{n=1}^{\infty} w_n$ is divergent the infinity, then we have

$$L^p \geq \inf_{n\in\mathbb{N}} \left[n^p - (n-1)^p\right] \frac{t_n}{w_n},$$

in which $t_n = \sum_{i=1}^{\infty} \tilde{w}_i \left(rt_{in} + st_{i+1,n}\right)^p$.

Theorem 18. *Let $H = (h_{nk})$ is the Hilbert matrix operator, $w_n = \frac{1}{n^{p+\alpha}}$ and $\tilde{w}_n = \frac{1}{n^{\alpha}}$ for every $n \in \mathbb{N}$, in which $p \geq 1$, $0 \leq p+\alpha \leq 1$ and $r \geq -s > 0$. For every decreasing sequences $x = (x_k)$ that are not negative terms, we have*

$$\|Hx\|_{p,\tilde{w},B} \geq L\|x\|_{p,w}$$

in which $L^p \geq \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}(i+1)^p(i+2)^p}$.

Proof. It is clear that both the Hilbert matrix $H = (h_{nk})$ and the sequence (w_n) fulfill the conditions listed in above Lemma, therefore, we obtain

$$\begin{aligned} L^p &\geq \inf_{n\in\mathbb{N}} \left[n^p - (n-1)^p\right] \frac{t_n}{w_n} \\ &\geq \inf_{n\in\mathbb{N}} n^{p-1} n^{p+\alpha} \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} \left(\frac{r}{i+n} + \frac{s}{i+n+1}\right)^p \\ &\geq \inf_{n\in\mathbb{N}} n^{2p+\alpha-1} \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} \left(\frac{r}{i+n} + \frac{s}{i+n+1}\right)^p. \end{aligned}$$

The rest of the proof can be derived in the same way as in the proof of Theorem 4.3 in Foroutannia (2019). □

6. Conclusion

In this manuscript, we have presented the norms for matrix operators which are defined between the weighted sequence space denoted by $\ell_p(w)$ and the weighted difference sequence space $\ell_p(\tilde{w}, B(r,s))$ which is valid for $1 \leq p < \infty$. To make the presentation more understandable, we have used some specific matrices like quasi summable ones (that is the transposes of Riesz and Cesàro matrices of the first order) and Hilbert matrix. Firstly, $\ell_p(\tilde{w}, B(r,s))$ space has been presented and its properties have been scrutinized. Next, we have tried to compute the lower bound for the matrix given from $\ell_p(w)$ into $\ell_p(\tilde{w}, B(r,s))$.

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