Functional Analysis
Volume 1, 2022, ID 8

FAC – SEAMS

On the Study of Continuous Generalized Frames With Bounded Operators in Hilbert Spaces

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ABSTRACT. This paper is devoted to the study of continuous $K-g-$frames which are extension of $K-g-$frames in Hilbert spaces. First, we give some property of continuous $K-g-$frames. Finally, we study the dual continuous $K-g-$bessel sequence of $K-g-$frames.

1. INTRODUCTION

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaffer [10] in 1952 to study some deep problems in nonharmonic Fourier series, after the fundamental paper [8] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames.

The concept of generalization of frames was proposed by G. Kaiser [14] and independently by Ali, Antoine and Gazeau [2] to a family indexed by some locally compact space endowed with a Radon measure. These frames are known as continuous frames. Gabrado and Han in [12] called these frames, frames associated with measurable spaces, Askari-Hemmat, Dehghan and Radjabalipour in [5] called them generalized frames and in mathematical physics they are referred to as Coherent states [3].

A continuous g-frames (or simply a c-g-frames) was firstly introduced by Abdollahpour and Faroughi in [1], it's an extension of g-frames and continuous frames. Recently, continuous g-frames in Hilbert spaces have been studied intensively.

For more on frames see [13, 15–19] and references therein.

We begin with a few preliminaries that will be needed. Let H, L be separable Hilbert spaces, (Ω, μ) a positive measure space. we denote by I_H the identity operator on H, $\{H_w\}_{w\in\Omega}$ a family of closed subspace of L, and $L(H, H_w)$ the set of all bounded linear operators from H into H_w. Suppose H_0 is a closed subspace of H, we define P_{H_0} is the

Date submitted: 2022-10-20.
2020 *Mathematics Subject Classification.* Primary 41A58; Secondary 42C15.
Key words and phrases. Frames, Continuous $g-$Frames, Continuous $K-g-$frames.

orthogonal projection of H into H_0. For $K \in L(H)$, the range and null space of K are denoted by $\mathcal{R}(K)$ and $\mathcal{N}(K)$, respectively. $l^2(\{H_w\}_{w\in\Omega})$ is defined by

$$l^2(\{H_w\}_{w\in\Omega}) = \{\{f_w\}_{w\in\Omega}, f_w \in H_w, w \in \Omega, \int_{w\in\Omega} \|f_w\|^2 d\mu(w) < \infty\}.$$

With the inner product given by

$$\langle \{f_w\}_{w\in\Omega}, \{g_w\}_{w\in\Omega}\rangle = \int_{w\in\Omega} \langle f_w, g_w\rangle d\mu(w).$$

It is clear that $l^2(\{H_w\}_{w\in\Omega})$ is a Hilbert space.

Definition 1.1. [1] We say that $\Lambda = \{\Lambda_w \in L(H, H_w), w \in \Omega\}$ is a continuous $g-$frame with respect to $\{H_w\}_{w\in\Omega}$ for H if

(1) for each $f \in H$, $\{\Lambda_w f\}_{w\in\Omega}$ is strongly measurable.
(2) there are two constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w) \leq B\|f\|^2, \quad \forall f \in H. \tag{1.1}$$

We call A, B lower and upper continuous $g-$frame bounds respectively. Λ is called a tight continuous $g-$frame if $A = B$, and a Parseval continuous $g-$frame if $A = B = 1$. A family Λ is called a continuous $g-$bessel sequence if the right hand inequality in (1.1) hold. In this case, B is called the bessel constant.

Proposition 1.2. *[1] Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $g-$frame with respect to $\{H_w\}_{w\in\Omega}$ for H with frame bounds A, B. Then, there exists a unique positive and invertible operator $S : H \to H$ such that for each $f, g \in H$*

$$\langle Sf, g\rangle = \int_{w\in\Omega} \langle f, \Lambda_w^* \Lambda_w g\rangle d\mu(w),$$

and $AI_H \leq S \leq BI_H$.

Definition 1.3. [4] Suppose that (Ω, μ) is a measure space with positive measure μ and $K \in L(H)$. A family $\Lambda = \{\Lambda_w \in L(H, H_w), w \in \Omega\}$ which $\{H_w\}_{w\in\Omega}$ is a family of Hilbert spaces, is called a continuous $K - g-$frame for H with respect to $\{H_w\}_{w\in\Omega}$ if

(1) for each $f \in H$, $\{\Lambda_w f\}_{w\in\Omega}$ is strongly measurable.
(2) there exist constants $0 < A \leq B < \infty$ such that

$$A\|K^* f\|^2 \leq \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w) \leq B\|f\|^2, \quad \forall f \in H. \tag{1.2}$$

The constants A, B are called lower and upper continuous $K - g-$frame bounds, respectively. If A, B can be chosen such that $A = B$, then Λ is called a tight continuous $K - g-$frame and if$A = B = 1$, it is called Parseval continuous $K - g-$frame. A family Λ is called a continuous $g-$Bessel sequence if the right hand inequality in (1.2) holds. In this case, B is called the Bessel constant.

Remark 1.4. It should be noted that the continuous $g-$frame operator S for continuous $K-g-$frame $\{\Lambda_w\}_{w\in\Omega}$ is not invertible in general, if K has closed range, then

$$S_\Lambda : \mathcal{R}(K) \to S(\mathcal{R}(K))$$

is invertible and self-adjoint.

Definition 1.5. [11] Suppose that (Ω, μ) is a measure space. A family of operators $\{\Lambda_w \in L(H, H_w), w \in \Omega\}$ is a continuous $g-$orthonormal basis for H with respect to $\{H_w\}_{w\in\Omega}$ if it satisfies the following:

(1) for all $f \in H$, $\{\Lambda_w f\}_{w\in\Omega}$ is strongly-measurable,
(2) for almost all $v \in \Omega$

$$\int_{w\in\Omega} \langle \Lambda_w^* f_w, \Lambda_v^* g_v \rangle d\mu(w) = \langle f_v, g_v \rangle,$$

(3) for each $f \in H$, $\int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w) = \|f\|^2$.

Definition 1.6. [6] Let H_1 and H_2 be tow Hilbert spaces with inner products $\langle .,. \rangle_{H_1}$, $\langle .,. \rangle_{H_2}$ and norms $\|.\|_{H_1}$, $\|.\|_{H_2}$, respectively, and let operator $T : H_1 \to H_2$. T is an isometry if $\|Tf\|_{H_2} = \|f\|_{H_1}$, for all $f \in H_1$. T is a co-isometry if its adjoint is an isometry.

Lemma 1.7. *[11] Let $\{\theta_w\}_{w\in\Omega}$ be a continuous $g-$orthonormal basis for H with respect to $\{H_w\}_{w\in\Omega}$ and $\{\Lambda_w \in L(H, H_w), w \in \Omega\}$ be a family such that $\{\Lambda_w f\}_{w\in\Omega}$ is strongly measurable for any $f \in H$. Then $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $g-$bessel family for H with respect to $\{H_w\}_{w\in\Omega}$ if and only if there exists a unique operator $T : H \to H$ such that $\Lambda_w = \theta_w T^*$, for almost all $w \in \Omega$.*

Definition 1.8. [11] Let $\{\theta_w\}_{w\in\Omega}$ be a continuous $g-$orthonormal basis for H with respect to $\{H_w\}_{w\in\Omega}$. The operator T in Lemma 1.7 is called the continuous $g-$preframe operator associated with $\{\Lambda_w\}_{w\in\Omega}$.

Lemma 1.9. *[11] Let $\{\theta_w\}_{w\in\Omega}$ be a continuous $g-$orthonormal basis and $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $g-$bessel family for H with repect to $\{H_w\}_{w\in\Omega}$. Suppose that T and S are the continuous $g-$preframe operator and continuous $g-$frame operator associated with $\{\Lambda_w\}_{w\in\Omega}$, repectively. Then $S = TT^*$.*

Lemma 1.10. *[9] Let $T_1 \in L(H_1, U)$ and $T_2 \in L(H_2, U)$. the following conditions are equivalent:*

(1) *$\mathcal{R}(T_1 \subset \mathcal{R}(T_2))$;*
(2) *There exists $\lambda > 0$ such that $T_1 T_2^* \leq \lambda T_2 T_2^*$;*
(3) *There exists a bounded operator $X \in L(H_1, H_2)$ such that $T_1 = T_2 X$.*

2. Some consrtuction of continuous $K-g-$frames

Definition 2.1. A sequence $\{a_w\}_{w\in\Omega}$ in $\mathbb{C}$ is said to be positively confined if

$$0 < \inf_{w\in\Omega} |a_w| \leq \sup_{w\in\Omega} |a_w| < \infty.$$

Now, we give a result about perturbations of $K-g-$frames.

Theorem 2.2. *Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K-g-$frame and $\Gamma_w \in L(H, H_w)$ for all $w \in \Omega$. If there exist constants $0 \le \alpha, \beta < \frac{1}{2}$, such that for every $f \in H$*

$$\int_{w\in\Omega} \|(a_w\Lambda_w - b_w\Gamma_w)f\|^2 d\mu(w) \le \alpha \int_{w\in\Omega} \|a_w\Lambda_w f\|^2 d\mu(w) + \beta \int_{w\in\Omega} \|b_w\Gamma_w f\|^2 d\mu(w),$$

then $\{\Gamma_w\}_{w\in\Omega}$ is a continuous $K-g-$frame, where $\{a_w\}_{w\in\Omega}$, $\{b_w\}_{w\in\}_{w\in\Omega}}$ are positively confined sequences.

Proof. Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K-g-$frame with bounds A and B. The parallelogram Law implies that, for each $f \in H$,

$$\begin{aligned}\int_{w\in\Omega} \|b_w\Gamma_w f\|^2 d\mu(w) &= \int_{w\in\Omega} \|b_w\Gamma_w f - a_w\Lambda_w f + a_w\Lambda_w f\|^2 d\mu(w)\\ &\le 2\Big(\int_{w\in\Omega} \|(a_w\Lambda_w - b_w\Gamma_w)f\|^2 d\mu(w) + \int_{w\in\Omega} \|a_w\Lambda_w f\|^2 d\mu(w)\Big)\\ &\le 2\Big(\alpha \int_{w\in\Omega} \|a_w\Lambda_w f\|^2 d\mu(w) + \beta \int_{w\in\Omega} \|b_w\Gamma_w f\|^2 d\mu(w)\\ &\qquad + \int_{w\in\Omega} \|a_w\Lambda_w f\|^2 d\mu(w)\Big),\end{aligned}$$

therefore,

$$(1-2\beta)\int_{w\in\Omega} |b_w|^2\|\Gamma_w f\|^2 d\mu(w) \le 2(\alpha+1)\int_{w\in\Omega} |a_w|^2\|\Lambda_w f\|^2 d\mu(w),$$

then,

$$\int_{w\in\Omega} \|\Gamma_w f\|^2 d\mu(w) \le \frac{2(\alpha+1)(\sup_{w\in\Omega}|a_w|)^2}{(1-2\beta)(\inf_{w\in\Omega}|b_w|)^2} B\|f\|^2.$$

For the same reason, we have

$$\begin{aligned}&\int_{w\in\Omega} \|a_w\Lambda_w f\|^2 d\mu(w)\\ &= \int_{w\in\Omega} \|a_w\Lambda_w f - b_w\Gamma_w f + b_w\Gamma_w f\|^2 d\mu(w)\\ &\le 2\Big(\alpha \int_{w\in\Omega} \|a_w\Lambda_w f\|^2 d\mu(w) + \beta \int_{w\in\Omega} \|b_w\Gamma_w f\|^2 d\mu(w) + \int_{w\in\Omega} \|b_w\Gamma_w f\|^2 d\mu(w\Big),\end{aligned}$$

hence,

$$(1-2\alpha)\int_{w\in\Omega} \|a_w\Lambda_w f\|^2 d\mu(w) \le 2(1+\beta)\int_{w\in\Omega} \|b_w\Gamma_w f\|^2 d\mu(w),$$

it follows that:

$$\int_{w\in\Omega} \|\Gamma_w f\|^2 d\mu(w) \ge \frac{(1-2\alpha)(\inf_{w\in\Omega}|a_w|)^2}{2(1+\beta)(\sup_{w\in\Omega}|b_w|)^2} A\|K^*f\|^2.$$

Therefore, $\{\Gamma_w\}_{w\in\Omega}$ is a continous $K-g-$frame. □

Corollary 2.3. *Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K-g-$frame and $\{\Gamma_w\}_{w\in\Omega} \in L(H, H_w)$ for all $w \in \Omega$. Then, the following statements hold*

(1) *If there exists* $0 < \alpha < \frac{1}{2}$, *such that for every* $f \in H$

$$\int_{w\in\Omega} \|(\Lambda_w - \Gamma_w)f\|^2 d\mu(w) \leq \alpha \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w),$$

then $\{\Gamma_w\}_{w\in\Omega}$ *is a continuous* $K-g-$*frame.*

(2) *If* $\sup_{w\in\Omega} \|\Lambda_w\| < \frac{\sqrt{2}}{2}$, *then the sequence* $\{\Lambda_w + \Lambda_w^2\}_{w\in\Omega}$ *is a continuous* $K-g-$*frame.*

In the sequel, we investigate the product of continuous $K-g-$frame and bounded linear operators as a continuous $K-g-$frame.

Theorem 2.4. *Let* $T \in L(H)$ *be a self-adjoint injective operator and* $TK = KT$. *Then,* $\{\Lambda_w\}_{w\in\Omega}$ *is a continuous* $K-g-$*frame if and only if* $\{\Lambda_w T\}_{w\in\Omega}$ *is a continuous* $K-g-$*frame.*

Proof. Let T be injective. Then, there exists $\tilde{T} \in L(H)$ with $T\tilde{T} = I_H$. Therefore $(\tilde{T})^* T^* K^* = K^*$. It follows that for every $f \in H$

$$\begin{aligned}\|K^* f\| &= \|\tilde{T}^* T^* K^* f\| \\ &\leq \|(\tilde{T})^*\| \|T^* K^* f\|.\end{aligned}$$

This shows that

$$\|(\tilde{T})^*\|^{-1} \|K^* f\| \leq \|T^* K^* f\|.$$

Assume that $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K-g-$frame. Then, there exist $A > 0$, such that for all $f \in H$,

$$\begin{aligned}\int_{w\in\Omega} \|\Lambda_w T f\|^2 d\mu(w) &\geq A\|K^* T f\|^2 \\ &= A\|K^* T^* f\|^2 \\ &= A\|T^* K^* f\|^2 \\ &\geq A\|(\tilde{T})^*\|^{-2} \|K^* f\|^2,\end{aligned}$$

hence, $\{\Lambda_w T\}_{w\in\Omega}$ is a continuous $K-g-$frame.

Conversely, let $\{\Lambda_w T\}_{w\in\Omega}$ be a continuous $K-g-$frame, since T is self-adjoint and injective, T is inversible, we have $T^{-1}K = KT^{-1}$ and $T^{-1}$is self-adjoint hence, $\{\Lambda_w\}_{w\in\Omega} = \{\Lambda_w T T^{-1}\}_{w\in\Omega}$ is a continuous $K-g-$frame. □

IT is well known from [7], if $T \in L(H)$, then $\mathcal{N}(T^*) = \mathcal{R}(T)^\perp$. Also, if T has closed range, then $\mathcal{R}(T) = \mathcal{N}(T^*)^\perp$.

Theorem 2.5. *Let* K *be surjective,* $T \in L(H)$, *and* $\{\Lambda_w T\}_{w\in\Omega}$ *be a continuous* $K-g-$*frame. Then, the following hold:*

(1) T *is injective.*

(2) *If* T *is self-adjoint and has closed range, then* T *is invertible and* $\{\Lambda_w\}_{w\in\}_{w\in\Omega}}$ *is a continuous* $T^{-1}K-g-$*frame.*

(3) *If* T *is self-adjoint and* $TK = KT$, *then* $\{\Lambda_w\}_{w\in\Omega}$ *is a continuous* $K-g-$*frame.*

Proof. (1) Assume that $\{\Lambda_w T\}_{w\in\Omega}$ is a continuous $K-g-$frame with bounds A and B. Then for every $f \in H$.

$$A\|K^* f\|^2 \leq \int_{w\in\Omega} \|\Lambda_w T f\|^2 d\mu(w) \leq B\|f\|^2,$$

hence, $\mathcal{N}(T)$ is a subspace of $\mathcal{N}(K^*)$,since K is surjective

$$\mathcal{N}(K^*) = \mathcal{R}(K)^\perp = \{0\}^\perp,$$

this shows that T is injective.

(2) We have T is injective. Since T is self-adjoint and has closed range

$$\mathcal{R}(T) = \mathcal{R}(T^*) = \mathcal{N}(T)^\perp = H.$$

It follows that T is surjective, therefore, T is invertible. Let $f \in H$, then, $Tg = f$ for some $g \in H$. Hence

$$\begin{aligned}\int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w) &= \int_{w\in\Omega} \|\Lambda_w T g\|^2 d\mu(w)\\ &\geq A\|K^* g\|^2\\ &= A\|K^* T^{-1} f\|^2\\ &= A\|(T^{-1}K)^* f\|^2.\end{aligned}$$

This implies that $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K-g-$frame.

(3) This follows from (1) and Theorem 2.4. □

In next result, we give conditions under which the sequence $\{\Lambda_w T_1 + \Gamma_w T_2\}_{w\in\Omega}$ is a $K-g-$frame, where $\{\Lambda_w\}_{w\in\Omega}$ and $\{\Gamma_w\}_{w\in\Omega}$ are continuous $K-g-$bessel sequences and $T_1, T_2 \in L(H)$.

Theorem 2.6. *Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K-g-$frame and $\Gamma_{ww\in\Omega}$ be a continuous $g-$bessel sequence. If for every $w \in \Omega$, $\mathcal{R}(\Lambda_w) \perp \mathcal{R}(\Gamma_w)$, then $\{\Lambda_w T_1 + \Gamma_w T_2\}_{w\in\Omega}$ is a continuous $(T_1^* K)-g-$frame, where $T_1, T_2 \in L(H)$.*

Proof. Let Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K-g-$frame and $\Gamma_{ww\in\Omega}$ be a continuous $g-$bessel sequence. THerefore, there exist constants A_1, B_1, B_2, such that for every $f \in H$

$$A_1\|K^* f\|^2 \leq \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w) \leq B_1\|f\|^2,$$

and

$$\int_{w\in\Omega} \|\Gamma_w f\|^2 d\mu(w) \leq B_2\|f\|^2.$$

Since $\mathcal{R}(\Lambda_w) \perp \mathcal{R}(\Gamma_w)$, we have $\mathcal{R}(\Lambda_w T_1) \perp \mathcal{R}(\Gamma_w T_2)$ for all $w \in \Omega$. Therefore we infer that

$$\|\Lambda_w T_1 f\| \leq \|(\Lambda_w T_1 + \Gamma_w T_2) f\|, \quad \forall f \in H.$$

Hence for each $f \in H$

$$\begin{aligned} A_1\|K^*T_1 f\|^2 &\le \int_{w\in\Omega} \|\Lambda_w T_1 f\|^2 d\mu(w) \\ &\le \int_{w\in\Omega} \|(\Lambda_w T_1 + \Gamma_w T_2)f\|^2 d\mu(w) \\ &= \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w) + \int_{w\in\Omega} \|\Gamma_w T_2 f\|^2 d\mu(w) \\ &\le B_1\|T_1 f\|^2 + B_2\|T_2 f\|^2 \\ &\le \Big(B_1\|T_1\|^2 + B_2\|T_2\|^2\Big)\|f\|^2, \end{aligned}$$

therefore, $\{\Lambda_w T_1 + \Gamma_w T_2\}_{w\in\Omega}$ is a continuous $(T_1^* K) - g-$frame. □

We conclude this section with the following result.

Corollary 2.7. *Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K - g-$frame and $\{\Gamma_w\}_{w\in\Omega}$ be a continuous $g-$bessel sequence, such that $\mathcal{R}(\Lambda_w) \perp \mathcal{R}(\Gamma_w)$ for all $w \in \Omega$. Then the following statements hold:*

(1) *The sequence $\{\Lambda_w T + \Gamma_w T\}_{w\in\Omega}$ is a continuous $(T^* K) - g-$frame.*
(2) *The sequences $\{\Lambda_w + \Gamma_w\}_{w\in\Omega}$ and $\{\Lambda_w - \Gamma_w\}_{w\in\Omega}$ are continuous $K - g-$frames.*
(3) *If $\{a_w\}w \in \Omega$ and $\{b_w\}_{w\in\Omega}$ are two positively confined sequences, then the sequence $\{a_w\Lambda_w + b_w\Gamma_w\}_{w\in\Omega}$ is a continuous $K - g-$frame.*

Proof. The statements (1) and (2) follow at once from Theorem 2.6. For (3), note that if $\{a_w\}_{w\in\Omega}$ and $\{b_w\}_{w\in\Omega}$ are positively confined, then $\{a_w\Lambda_w\}_{w\in\Omega}$ and $\{b_w\Gamma_w\}_{w\in\Omega}$ are continuous $K - g-$frames. Since $\mathcal{R}(\Lambda_w) \perp \mathcal{R}(\Gamma_w)$, we have $\mathcal{R}(a_w\Lambda_w) \perp \mathcal{R}(b_w\Gamma_w)$ for all $w \in \Omega$. Now apply (2). □

3. Characterizing continuous $K - g-$frames by quotient maps

Let $T_1, T_2 \in L(H)$. The map $[T_1/T_2] : \mathcal{R}(T_2) \to \mathcal{R}(T_1)$ defined by $T_2 f \to T_2 f$ is called the quotient map. It is proved that $[T_1/T_2]$ is a linear operator on H if and only if $\mathcal{N}(T_2) \subseteq \mathcal{N}(T_1)$.

Theorem 3.1. *Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $g-$bessel sequence with the frame operator S and $K \in L(H)$. THen, $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K - g-$frame if and only if the quotient operator $[K^*/S^{\frac{1}{2}}]$ is a bounded linear operator. In this case, $K = S^{\frac{1}{2}}X$ for some $X \in L(H)$.*

Proof. First, note that for every $f \in H$, we have

$$\begin{aligned} \|S^{\frac{1}{2}} f\| &= \langle Sf, f\rangle \\ &= \int_{w\in\Omega} \langle \Lambda_w^*\Lambda_w f, f\rangle d\mu(w) \\ &= \int_{w\in\Omega} \langle \Lambda_w f, \Lambda_w f\rangle d\mu(w) \\ &= \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w). \end{aligned} \tag{3.1}$$

Now, let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K-g-$frame. Then exists constant $A>0$, such that

$$A\|K^*f\|^2 \leq \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w), \quad \forall f \in H.$$

From this and (3.1), we obtain

$$\|S^{\frac{1}{2}}f\|^2 \geq A\|K^*f\|^2,$$

hence, $\mathcal{N}(S^{\frac{1}{2}}) \subseteq \mathcal{N}(K^*)$, which implies that the quotient map

$[K^*/S^{\frac{1}{2}}] : \mathcal{R}(S^{\frac{1}{2}}) \to \mathcal{R}(K^*)$ defined by $[K^*/S^{\frac{1}{2}}](S^{\frac{1}{2}}f) = K^*f$ is a bounded linear operator.

Conversely, assume that $[K^*/S^{\frac{1}{2}}]$ is a bounded linear operator Then, there exists $c>0$, such that for all $f\in H$

$$\|K^*f\|^2 \leq c\|S^{\frac{1}{2}}f\|^2.$$

From this and (3.1), we infer that

$$\|K^*f\|^2 \leq c\int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w),$$

therefore, $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K-g-$frame.

To complete the proof, let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K-g-$frame. then

$$S \geq \alpha KK^* \quad for \quad some \quad \alpha > 0.$$

By Lemma 1.10 there exists $X \in L(H)$, such that $K = S^{\frac{1}{2}}X$. □

Corollary 3.2. *Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K-g-$frame and $n\in\mathbb{N}$. Then , $[K^*/S^{\frac{1}{2}}]$ is a continuous K^n-g-frame. If K is invertible, then the converse holds.*

Proof. Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K-g-$frame and $n\in\mathbb{N}$. Then $[K^*/S^{\frac{1}{2}}]$ is a bounded linear by Theorem 3.1. Hence, there exists $C>0$, such that for every $f\in H$

$$\|K^*f\| \leq C\|S^{\frac{1}{2}}f\|.$$

Therefore, for every $f\in H$ we have

$$\begin{aligned}\|(K^{n-1})^*(K^*)f\| &\leq \|(K^{n-1})^*\|\|K^*f\| \\ &\leq C\|(K^{n-1})^*\|\|S^{\frac{1}{2}}f\|.\end{aligned}$$

This shows that $[(K^n)^*/S^{\frac{1}{2}}]$ is a bounded linear operator. That is, $\{\Lambda_w\}_{w\in\Omega}$ is a continuous K^n-g-frame.

for the converse, assume that $\{\Lambda_w\}_{w\in\Omega}$ is a continuous K^n-g-frame. Note that if K is invertible, then

$$K = K^nK^{1-n}.$$

It follows from Lemma 1.10 that:

$$KK^* \leq \alpha K^n(K^n)^*.$$

for some $\alpha > 0$. Therefore, for every $f \in H$, we have

$$\begin{aligned}
\|K^* f\|^2 &= \langle K^* f, K^* f\rangle \\
&= \langle KK^* f, f\rangle \\
&\leq \langle \alpha K^n (K^n)^* f, f\rangle \\
&= \alpha\langle (K^n)^* f, (K^n)^* f\rangle \\
&= \alpha\|(K^n)^* f\|^2.
\end{aligned}$$

This implies that the quotient operator $[K^*/S^{\frac{1}{2}}]$ is bounded. Thus, $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K - g-$frame. □

Theorem 3.3. *Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $K - g-$ frame with frame operator S. Then the following assertion are equivalent*

(1) $\{\Lambda_w T\}_{w\in\Omega}$ *is a continuous $TK - g-$frame.*
(2) $[(TK)^*/(S^{\frac{1}{2}}T)]$ *is bounded.*
(3) $[(TK)^*/(T^*ST)^{\frac{1}{2}}]$ *is bounded.*

Proof. For every $f \in H$, we have

$$\begin{aligned}
\int_{w\in\Omega} (\Lambda_w T)^*(\Lambda_w T) f d\mu(w) &= \int_{w\in\Omega} T^*\Lambda_w^*\Lambda_w T f d\mu(w) \\
&= T^*\left(\int_{w\in\Omega} \Lambda_w^*\Lambda_w T f d\mu(w)\right) \\
&= T^*STf.
\end{aligned}$$

Hence, the frame operator of $\{\Lambda_w T\}_{w\in\Omega}$ is T^*ST. Now, Theorem 3.1 shows that (1) and (3) are equivalent.

for every $f \in H$, we have

$$\begin{aligned}
\|(T^*ST)^{\frac{1}{2}} f\|^2 &= \langle T^*ST)^{\frac{1}{2}} f, T^*ST)^{\frac{1}{2}} f\rangle \\
&= \langle (T^*ST) f, f\rangle \\
&= \langle STf, Tf\rangle \\
&= \langle S^{\frac{1}{2}}Tf, S^{\frac{1}{2}}Tf \\
&= \|S^{\frac{1}{2}}Tf\|^2.
\end{aligned}$$

Therefore, (2) and (3) are equivalent. □

Corollary 3.4. *Let $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $g-$frame. Then, the assertions are equivalent.*

(1) $\{\Lambda_w K\}_{w\in\Omega}$ *is a continuous $K - g-$frame.*
(2) $[K^*/(S^{\frac{1}{2}}K)]$ *is a bounded.*

4. DUAL OF CONTINUOUS $K - g-$BESSEL SEQUENCE

Definition 4.1. Suppose that $K \in L(H)$ and $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K - g-$frame for H with respect to $\{H_w\}_{w\in\Omega}$. A $g-$bessel sequence $\{\Gamma_w\}_{w\in\Omega}$ for H with respect to

$\{H_w\}_{w\in\Omega}$ is said to be a dual continuous $K-g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$ if

$$Kf = \int_{w\in\Omega} \Lambda_w^* \Gamma_w f d\mu(w), \quad \forall f \in H$$

Theorem 4.2. *Let $K \in L(H)$ and $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $g-$bessel sequence for H with respect to $\{H_w\}_{w\in\Omega}$. T is the continuous $g-$preframe operator associated with $\{\Lambda_w\}_{w\in\Omega}$, then $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K-g-$frame if and only if $\mathcal{R}(K) \subset \mathcal{R}(T)$.*

Proof. Suppose that $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K-g-$frame with respect to $\{H_w\}_{w\in\Omega}$. Then there exists a constant $A > 0$ such that

$$A\|K^*f\|^2 \leq \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w), \quad \forall f \in H.$$

Let $\{\theta_w \in L(H, H_w), w \in \Omega\}$ is the continuous $g-$orthonormal basis for H with respect to $\{H_w\}_{w\in\Omega}$. Then, by Lemma 1.7, we have $\Lambda_w = \theta_w T^*$, $\forall w \in \Omega$, so

$$A\|K^*f\|^2 \leq \int_{w\in\Omega} \|\theta_w T^* f\|^2 d\mu(w) = \|T^*f\|^2, \quad \forall f \in H,$$

hence, $AKK^* \leq TT^*$. Therefore by Lemma 1.10, we have $\mathcal{R}(K) \subset \mathcal{R}(T)$.

Conversely, assume that $\mathcal{R}(K) \subset \mathcal{R}(T)$. By Lemma 1.10, there exists a constant $\lambda > 0$ such that $KK^* \leq \lambda TT^*$, then

$$\langle \frac{1}{\lambda} KK^* f, f\rangle \leq \langle TT^* f, f\rangle,$$

so,

$$\frac{1}{\lambda}\|K^*f\|^2 \leq \|T^*f\|^2.$$

Note $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $g-$bessel sequence and suppose that $\{\theta_w\}_{w\in\Omega}$ is the continuous $g-$orthonormal basis for H with respect to $\{H_w\}_{w\in\Omega}$, then by Lemma 1.7, we have

$$\|T^*f\|^2 = \int_{w\in\Omega} \|\theta_w T^* f\|^2 d\mu(w) = \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w),$$

hence,

$$\frac{1}{\lambda}\|K^*f\|^2 \leq \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w),$$

so, the continuous $g-$bessel sequence is a continuous $K-g-$frame. □

Theorem 4.3. *Let $K \in L(H)$ and $\{\Lambda_w\}_{w\in\Omega}$ be a continuous $g-$bessel sequence for H with respect to $\{H_w\}_{w\in\Omega}$. The associated continuous $g-$preframe operator with $\{\Lambda_w\}_{w\in\Omega}$ is T and $\{\theta_w\}_{w\in\Omega}$ is the continuous $g-$orthonormal basis for H with respect to $\{H_w\}_{w\in\Omega}$. Then T is a co-isometry if and only if $\{\Lambda_w K^*\}_{w\in\Omega}$ is a continuous Parseval $K-g-$frame.*

Proof. From the definition of continuous $g-$orthonormal basis, we have

$$\int_{w\in\Omega} \|\Lambda_w K^* f\|^2 d\mu(w) = \int_{w\in\Omega} \|\theta_w T^* K^* f\|^2 d\mu(w) = \|T^*K^*f\|^2, \quad \forall f \in H.$$

Which implies the conclusion is obvious. □

Theorem 4.4. *Let $K \in L(H)$ and $\{\theta_w\}_{w\in\Omega}$ be a continuous $g-$orthonormal basis for H with respect to $\{H_w\}_{w\in\Omega}$. $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K-g-$frame for H with respect to $\{H_w\}_{w\in\Omega}$ with the continuous $g-$preframe operator T. U is the continuous $g-$preframe operator of continuous $g-$bessel sequence $\{\Gamma_w\}_{w\in\Omega}$. If U is invertible and U^{-1} is the right inverse of T, then $\{\Gamma_w\}_{w\in\Omega}$ is a continuous $K-g-$frame.*

Proof. $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K-g-$frame for H with respect to $\{H_w\}_{w\in\Omega}$, then there exists a constant $A > 0$ such that

$$A\|K^*f\|^2 \leq \int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w),$$

and we have $\{\theta_w\}_{w\in\Omega}$ is a continuous $g-$orthonormal basis for H with respect to $\{H_w\}_{w\in\Omega}$, so

$$\int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w) = \int_{w\in\Omega} \|\theta_w T^* f\|^2 d\mu(w), \quad \forall f \in H,$$

Since U is invertible and $TU^{-1} = I_H$, we obtain

$$\int_{w\in\Omega} \|\Lambda_w f\|^2 d\mu(w) = \int_{w\in\Omega} \|\theta_w U^*(U^*)^{-1} T^* f\|^2 d\mu(w) = \int_{w\in\Omega} \|\theta_w U^* f\|^2 d\mu(w), \quad \forall f \in H,$$

we have U is the continuous $g-$preframe operator of continuous $g-$bessel sequence $\{\Gamma_w\}_{w\in\Omega}$, then

$$A\|K^*f\|^2 \leq \int_{w\in\Omega} \|\Gamma_w f\|^2 d\mu(w), \quad \forall f \in H.$$

□

Theorem 4.5. *Let $K \in L(H)$ and $\{\theta_w\}_{w\in\Omega}$ be a continuous $g-$orthonormal basis for H with respect to $\{H_w\}_{w\in\Omega}$. $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K-g-$frame for H with respect $\{H_w\}_{w\in\Omega}$ with the continuous $g-$preframe operator T. U is the continuous $g-$preframe operator of continuous $g-$bessel sequence $\{\Gamma_w\}_{w\in\Omega}$. Then $\{\Gamma_w\}_{w\in\Omega}$ is the dual continuous $K-g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$ if and only if $K = TU^*$.*

Proof. Assume that $\{\Gamma_w\}_{w\in\Omega}$ is the dual continuous $K-g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$, then for each $f \in H$

$$Kf = \int_{w\in\Omega} \Lambda_w^* \Gamma_w f d\mu(w).$$

Since $\{\theta_w\}_{w\in\Omega}$ is the continuous $g-$orthonormal basis for H, then by Lemma 1.7, we have for all $f \in H$

$$\begin{aligned} Kf &= \int_{w\in\Omega} (\theta_w T^*)^*(\theta_w U^*) f d\mu(w) \\ &= T\int_{w\in\Omega} \theta_w^* \theta_w U^* f d\mu(w) \\ &= TU^* f, \end{aligned}$$

so, $K = TU^*$.

Conversely, suppose that $K = TU^*$. We have

$$\Lambda_w = \theta_w T^*, \quad \Gamma_w = \theta_w U^*, \quad \forall w \in \Omega.$$

Hence, for each $f \in H$

$$\begin{aligned}\int_{w\in\Omega} \Lambda_w^* \Gamma_w f d\mu(w) &= \int_{w\in\Omega} (\theta_w T^*)^* (\theta_w U^*) f d\mu(w) \\ &= T \int_{w\in\Omega} \theta_w^* \theta_w U^* f d\mu(w) \\ &= TU^* f \\ &= Kf.\end{aligned}$$

This shows that $\{\Gamma_w\}_{w\in\Omega}$ is the dual continuous $K-g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$. □

Theorem 4.6. *Let $K \in L(H)$. $\{\Gamma_w\}_{w\in\Omega}$ is the dual continuous $K-g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$. Suppose that $V \in L(H)$, if V is a co-isometry then $\{V^*\Gamma_w\}_{w\in\Omega}$ is the dual continuous $K-g-$bessel sequence of $\{V^*\Lambda_w\}_{w\in\Omega}$.*

Proof. Since V is a co-isometry, then $VV^* = I_H$. So, for each $f \in H$

$$\begin{aligned}\int_{w\in\Omega} (V^*\Lambda_w)^* (V^*\Gamma_w) f d\mu(w) &= \int_{w\in\Omega} \Lambda_w^* VV^* \Gamma_w f d\mu(w) \\ &= \int_{w\in\Omega} \Lambda_w^* \Gamma_w f d\mu(w) \\ &= Kf.\end{aligned}$$

□

Theorem 4.7. *Let $K \in L(H)$ be with closed range and $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K-g-$frame for H with respect to $\{H_w\}_{w\in\Omega}$. Then $\{\Lambda_w P_{S(\mathcal{R}(K))}(S_\Lambda^{-1})^* K\}_{w\in\Omega}$ is the dual continuous $K-g-$bessel sequence of $\{\Lambda_w P_{\mathcal{R}(K)}\}_{w\in\Omega}$.*

Proof. It is easy to check that $\{\Lambda_w P_{S(\mathcal{R}(K))}(S_\Lambda^{-1})^* K\}_{w\in\Omega}$ is continuous $g-$bessel sequence. Since S_Λ is self-adjoint and invertible, we have for each $f \in H$

$$\begin{aligned}Kf &= (S_\Lambda^{-1} S_\Lambda)^* Kf \\ &= S_\Lambda^* (S_\Lambda^{-1})^* Kf \\ &= S_\Lambda^* P_{S(\mathcal{R}(K))} (S_\Lambda^{-1})^* Kf \\ &= P_{\mathcal{R}(K)} S_\Lambda^* P_{S(\mathcal{R}(K))} (S_\Lambda^{-1})^* Kf \\ &= P_{\mathcal{R}(K)} \int_{w\in\Omega} \Lambda_w^* \Lambda_w P_{S(\mathcal{R}(K))} (S_\Lambda^{-1})^* Kf d\mu(w) \\ &= \int_{w\in\Omega} (\Lambda_w P_{\mathcal{R}(K)})^* (\Lambda_w P_{S(\mathcal{R}(K))} (S_\Lambda^{-1})^* K) f d\mu(w).\end{aligned}$$

□

Theorem 4.8. *Let $K \in L(H)$ be with closed range and $\{\Lambda_w\}_{w\in\Omega}$ is a continuous $K-g-$frame for H with respect to $\{H_w\}_{w\in\Omega}$. Then $\{\phi_w\}_{w\in\Omega}$ is the dual continuous $K-g-$bessel sequence of $\{\Lambda_w P_{\mathcal{R}(K)}\}_{w\in\Omega}$ if and only if $\forall w \in \Omega$, $\phi_w = \Gamma_w + \theta_w \psi$, where $\Gamma_w = \Lambda_w P_{S(\mathcal{R}(K))}(S_\Lambda^{-1})^* K$ and $\{\theta_w\}_{w\in\Omega}$ is the continuous $g-$orthonormal basis of H with respect to $\{H_w\}_{w\in\Omega}$, $\psi \in L(H)$ and T is the continuous $g-$preframe operator of $\{\Lambda_w\}_{w\in\Omega}$ such that $P_{\mathcal{R}(K)} T\psi = 0$.*

Proof. ($\Longleftarrow$) Define the operator $\psi = U^* - T^*P_{S(\mathcal{R}(K))}(S_\Lambda^{-1})^*K$, where $U \in L(H)$ is the continuous $g-$preframe operator associated with $\{\phi_w\}_{w\in\Omega}$, then $\psi \in L(H)$. By Theorem 4.5 and Lemma 1.9, we know $TU^* = K$ and $S = TT^*$, hence

$$\begin{aligned} P_{\mathcal{R}(K)}T\psi f &= P_{\mathcal{R}(K)}TU^*f - P_{\mathcal{R}(K)}TT^*P_{S(\mathcal{R}(K))}(S_\Lambda^{-1})^*Kf \\ &= Kf - (S_\Lambda)^*(S_\Lambda^{-1})^*Kf \\ &= 0 \end{aligned}$$

Moreover,

$$\theta_w\psi = \theta_wU^* - \theta_wT^*P_{S(\mathcal{R}(K))}(S_\Lambda^{-1})^*K = \phi_w - \Lambda_wP_{S(\mathcal{R}(K))}(S_\Lambda^{-1})^*K,$$

hence

$$\Gamma_w + \theta_w\psi = \Lambda_wP_{S(\mathcal{R}(K))}(S_\Lambda^{-1})^*K + \phi_w - \Lambda_wP_{S(\mathcal{R}(K))}(S_\Lambda^{-1})^*K = \phi_w.$$

($\Longrightarrow$) Assume that $\psi \in L(H)$ and $P_{\mathcal{R}(K)}T\psi = 0$, then it is obvious that $\{\phi_w\}_{w\in\Omega} = \{\Gamma_w + \theta_w\psi\}_{w\in\Omega}$ is a continuous $g-$bessel sequence. So,

$$\begin{aligned} \int_{w\in\Omega}(\Lambda_wP_{\mathcal{R}(K)})^*\phi_wfd\mu(w) &= \int_{w\in\Omega}(\Lambda_wP_{\mathcal{R}(K)})^*\Gamma_wfd\mu(w) + \int_{w\in\Omega}(\Lambda_wP_{\mathcal{R}(K)})^*\theta_w\psi fd\mu(w) \\ &= Kf + P_{\mathcal{R}(K)}\int_{w\in\Omega}T\theta_w^*\theta_w\psi fd\mu(w) \\ &= Kf + P_{\mathcal{R}(K)}T\psi f \\ &= Kf, \end{aligned}$$

we conclude that $\{\phi_w\}_{w\in\Omega}$ is the dual continuous $K - g-$bessel sequence of $\{\Lambda_wP_{\mathcal{R}(K)}\}_{w\in\Omega}$. □

Theorem 4.9. *Let $K \in L(H)$ and $\{\Gamma_w\}_{w\in\Omega}$ be the dual continuous $K - g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$ whose continuous $g-$preframe operator T. Suppose the continuous $g-$preframe operator of the continuous $g-$bessel sequence $\{\phi_w\}_{w\in\Omega}$ is U, then $TU^* = 0$ if and only if $\{\Gamma_w + \phi_w\}_{w\in\Omega}$ is the dual continuous $K - g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$.*

Proof. Assume that $TU^* = 0$ and $\{\theta_w\}_{w\in\Omega}$ is the continuous $g-$orthonormal basis for H with respect to $\{H_w\}_{w\in\Omega}$, then

$$\begin{aligned} \int_{w\in\Omega}\Lambda_w^*\phi_wfd\mu(w) &= \int_{w\in\Omega}(\theta_wT^*)^*(\theta_wU^*)fd\mu(w) \\ &= T\int_{w\in\Omega}\theta_w^*\theta_wU^*f \\ &= TU^*f \\ &= 0. \end{aligned}$$

Hence, for each $f \in H$,

$$\int_{w\in\Omega}\Lambda_w^*(\Gamma_w + \phi_w)fd\mu(w) = \int_{w\in\Omega}\Lambda_w^*\Gamma_wfd\mu(w) = Kf.$$

For the other implication, just put the above calculation process contrary again and we can get the result. □

Theorem 4.10. *let $K \in L(H)$. $\{\Gamma_w\}_{w\in\Omega}$ and $\{\phi_w\}_{w\in\Omega}$ are both the dual continuous $K-g-$bessel sequences of $\{\Lambda_w\}_{w\in\Omega}$, respectively. Operators V_1 and V_2 are two linear operators on H, if $V_1 + V_2 = I_H$ then $\{\Gamma_w V_1 + \phi_w V_2\}_{w\in\Omega}$ is the dual continuous $K - g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$.*

Proof. we have for each $f \in H$,

$$\begin{aligned}\int_{w\in\Omega} \Lambda_w^*(\Gamma_w V_1 + \phi_w V_2) f d\mu(w) &= \int_{w\in\Omega} \Lambda_w^* \Gamma_w V_1 f d\mu(w) + \int_{w\in\Omega} \Lambda_w^* \phi_w V_2 f d\mu(w)\\ &= KV_1 + KV_2 f\\ &= K(V_1 + V_2) f\\ &= Kf.\end{aligned}$$

□

Corollary 4.11. *Let $K \in L(H)$ and it is invertible. $\{\Gamma_w\}_{w\in\Omega}$ and $\{\phi_w\}_{w\in\Omega}$ are both the dual continuous $K - g-$bessel sequences of $\{\Lambda_w\}_{w\in\Omega}$, respectively. Operators V_1 and V_2 are two linear operators on H, then $\{\Gamma_w V_1 + \phi_w V_2\}_{w\in\Omega}$ is the dual continuous $K - g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$ if and only if $V_1 + V_2 = I_H$.*

Proof. ($\Longleftarrow$) Suppose that $V_1+V_2 = I_H$, then by Theorem 4.10, we have $\{\Gamma_w V_1+\phi_w V_2\}_{w\in\Omega}$ is the dual continuous $K - g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$.

($\Longrightarrow$) Assume that $\{\Gamma_w V_1 + \phi_w V_2\}_{w\in\Omega}$ is the dual continuous $K - g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$, so for each $f \in H$,

$$\begin{aligned}Kf &= \int_{w\in\Omega} \Lambda_w^*(\Gamma_w V_1 + \phi_w V_2) f d\mu(w)\\ &= \int_{w\in\Omega} \Lambda_w^* \Gamma_w V_1 f d\mu(w) + \int_{w\in\Omega} \Lambda_w^* \phi_w V_2 f d\mu(w)\\ &= KV_1 f + KV_2 f\\ &= K(V_1 + V_2) f.\end{aligned}$$

Hence,

$$K = K(V_1 + V_2),$$

Since K is invertible and we conclude that $V_1 + V_2 = I_H$. □

Theorem 4.12. *Let $K \in L(H)$ anve closed range and $\{\Gamma_w\}_{w\in\Omega}$ be the dual continuous $K - g-$bessel sequence of $\{\Lambda_w\}_{w\in\Omega}$. Suppose that α is a complex number and $\mathcal{R}(K) \subset S(\mathcal{R}(K))$, then the sequence $\{\Delta_w\}_{w\in\Omega}$ defined by*

$$\Delta_w = \alpha\Gamma_w + (1 - \alpha)\Lambda_w S_\Lambda^{-1} K \tag{4.1}$$

is a dual continuous $K - g-$bessel sequance of $\{\Lambda_w\}_{w\in\Omega}$ for H with respect to $\{H_w\}_{w\in\Omega}$.

Proof. Since $\{\Lambda_w\}_{w\in\Omega}$ and $\{\Gamma_w\}_{w\in\Omega}$ are both continuous $g-$bessel sequences, then it is easy to check the sequence $\{\Delta_w\}_{w\in\Omega}$ is also a continuous $g-$bessel sequence. Hence, for each $f \in H$,

$$\begin{aligned}\int_{w\in\Omega} \Lambda_w^* \Delta_w f d\mu(w) &= \int_{w\in\Omega} \Lambda_w^* \alpha \Gamma_w f d\mu(w) + \int_{w\in\Omega} \Lambda_w^* (1-\alpha) \Lambda_w S_\Lambda^{-1} K f d\mu(w) \\ &= \alpha K f + (1-\alpha) \int_{w\in\Omega} \Lambda_w^* \Lambda_w S_\Lambda^{-1} K f d\mu(w) \\ &= \alpha K f + (1-\alpha) K f \\ &= K f.\end{aligned}$$

□

5. Acknowledgments

It is our great pleasure to thank the referee for his careful reading of the paper and for several helpful suggestions.

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