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Fixed Point Theorems in C^*-Algebra Valued Extended Hexagonal b-Asymmetric Metric Spaces

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ABSTRACT. In this present paper we introduce the class of C^*-algebra valued extended hexagonal b- asymmetric metric spaces and establish certain fixed point theorem. Non-trivial examples are further provided to support the hypotheses of our results.

1. Introduction

Fixed point theory is an important tool for solving existence of solutions of many non-linear problems in various branches of science and has been studied in differents spaces.

Ma et al. [13] introduced the notion of C^*-algebra valued metric spaces by replacing the set of real numbers by the set of all positive elements of a unital C^*- algebra.

In 2015, Ma and Jiang [14] introduced a concept of C^*-algebra valued $b-$ metric spaces which generalize an ordinary C^*-algebra valued space and give some fixed point theorems.

In 2017 Kamran et al [7] initiated the concept of extented $b-$ metric spaces.

Definition 1.1. [7] Let X be a non empty set and $E : X \times X \to [1, \infty[$. A function $d : X \times X \to [0, \infty[$ is called an extended $b-$ metric if it satisfies:

(1) $d(x, y) = 0 \Leftrightarrow x = y \; \forall x, y \in X$.
(2) $d(x, y) = d(y, x), \forall x, y \in X$.
(3) $d(x, y) \leq E(x, y)[d(x, z) + d(z, y)] \; \forall x, y, z \in X$.

(X, d) is called an extended $b-$ metric space.

The notion of extended hexagonal $b-$ metric spaces was introduced by Kalpana et al. [5].

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Definition 1.2. [5] Let X be a non empty set and $E : X \times X \to [1, \infty[$. A function $d : X \times X \to [0, \infty[$ is called an extended hexagonal $b-$ metric if it satisfies:

(1) $d(x, y) = 0 \Leftrightarrow x = y \; \forall x, y \in X$.
(2) $d(x, y) = d(y, x); \forall x, y \in X$.
(3) $d(x, y) \leq E(x, y)[d(x, u) + d(u, v) + d(v, w) + d(w, z) + d(z, y)] \; \forall x, y, u, v, w, z \in X$ and $x \neq u, u \neq v, v \neq w, w \neq z, z \neq y$.

(X, d) is called an extended hexagonal $b-$ metric space.

2. PRELIMINARIES

Throughout this paper, we denote $\mathbb{A}$ by an unital (i.e ,unity element I) C^*-algebra with linear involution $*$, such that for all $x, y \in \mathbb{A}$,

$$(xy)^* = y^* x^*, \text{and } x^{**} = x.$$

We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$.

If $x \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x)$ is the spectrum of x. Using positive element, we can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows :

$$x \preceq y \text{ if and only if } y - x \succeq \theta,$$

where θ means the zero element in $\mathbb{A}$.

We denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ by $\mathbb{A}_+$ and $|x| = (x^* x)^{\frac{1}{2}}$, $\mathbb{A}'$ will denote the set $\{a \in \mathbb{A}_+; ab = ba, \forall b \in \mathbb{A}\}$ and $\mathbb{A}_I^{'} = \{a \in \mathbb{A}; ab = ba, \forall b \in \mathbb{A} \;\; and \; a \succeq I\}$.

Lemma 2.1. *[15] Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I.*

(1) *For any $x \in \mathbb{A}_+$ we have $x \preceq I \iff \|x\| \leq 1$.*
(2) *If $a \in \mathbb{A}_+$ with $\|a\| < \dfrac{1}{2}$ then $I - a$ is invertible and $\|a(1 - a)^{-1}\| < 1$.*
(3) *Suppose that $a, b \in \mathbb{A}_+$ and $ab = ba$, then $ab \succeq \theta$.*
(4) *Let $a \in \mathbb{A}' = \{a \in \mathbb{A}; ab = ba \; \forall b \in \mathbb{A}\}$, if $b, c \in \mathbb{A}$, with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}'_+$ is invertible operator, then $(I - a)^{-1} b \succeq (I - a)^{-1} c$.*

Recently, Asim et al. [3] developed a concept of C^*-algebra valued extended $b-$ metric spaces.

Definition 2.2. Let X be a non empty set and $E : X \times X \to \mathbb{A}_I^{'}$. A function $d : X \times X \to \mathbb{A}$ is called a C^*-algebra valued extended $b-$ metric spaces on X if it satisfies:

(1) $d(x, y) = \theta \Leftrightarrow x = y \; \forall x, y \in X$ and $d(x, y) \succeq \theta$.
(2) $d(x, y) = d(y, x) \; \forall x, y \in X$.
(3) $d(x, y) \preceq E(x, y)[d(x, z) + d(z, y)] \; \forall x, y, z \in X$.

$(X, \mathbb{A}, d)$ is called a C^*-algebra valued extended $b-$ metric space.

Later Kalpana et al. [6] defined in the following C^*-algebra valued hexagonal $b-$ metric spaces.

Definition 2.3. Let X be a non empty set and $b \in \mathbb{A}_I^{'}$ such that $b \succeq I$.

Suppose the mapping $d : X \times X \to \mathbb{A}$ if it satisfies:

(1) $d(x, y) \succeq \theta$ and $d(x, y) = \theta \Leftrightarrow x = y \; \forall x, y \in X$.

(2) $d(x, y) = d(y, x); \forall x, y \in X$.
(3) $d(x, y) \preceq b[d(x, u) + d(u, v) + d(v, w) + d(w, z) + d(z, y)]$ $\forall x, y, u, v, w, z \in X$ and $x \neq u, u \neq v, v \neq w, w \neq z, z \neq y$.

d is called a C^*-algebra valued hexagonal $b-$ metric and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued hexagonal $b-$ metric space.

The definition of C^*-algebra valued extended hexagonal $b-$ metric space was defined in the following way in [5].

Definition 2.4. Let X be a non empty set and $E : X \times X \to \mathbb{A}'_I$.
Suppose the mapping $d : X \times X \to \mathbb{A}$ if it satisfies:

1) $d(x, y) \succeq \theta$ and $d(x, y) = \theta \Leftrightarrow x = y$; $\forall x, y \in X$.
2) $d(x, y) = d(y, x); \forall x, y \in X$.
3) $d(x, y) \preceq E(x, y)[d(x, u) + d(u, v) + d(v, w) + d(w, z) + d(z, y)]$ $\forall x, y, u, v, w, z \in X$ and $x \neq u, u \neq v, v \neq w, w \neq z, z \neq y$.

$(X, \mathbb{A}, d)$ is called a C^*-algebra valued extended hexagonal $b-$ metric space.

Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, asymmetric metric space were introduce by Wilson [17] as metric spaces, but without the requirement that the asymmetric metric d has to satisfy $d(x, y) = d(y, x)$. For further investigations on the concept of asymmetric metric, the readers can view [2, 8–12, 16].

Influenced by all the above concepts, we introduce the class of C^*-algebra valued hexagonal $b-$ asymmetric metric spaces and C^*-algebra valued extended hexagonal $b-$ asymmetric metric space and establish certain fixed point theorems.

3. Main result

Definition 3.1. Let X be a non empty set and $b \in \mathbb{A}'_I$ such that $b \succeq I$.
Suppose the mapping $d : X \times X \to \mathbb{A}$ if it satisfies:

(1) $d(x, y) \succeq \theta$ and $d(x, y) = \theta \Leftrightarrow x = y$ $\forall x, y \in X$.
(2) $d(x, y) \preceq b[d(x, u) + d(u, v) + d(v, w) + d(w, z) + d(z, y)]$ $\forall x, y, u, v, w, z \in X$ and $x \neq u, u \neq v, v \neq w, w \neq z, z \neq y$.

d is called a C^*-algebra valued hexagonal $b-$ metric and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued hexagonal $b-$asymmetric metric space.

Example 3.2. Let $\mathbb{A} = \mathbb{R}^2$ a C^*-algebra with the partial order

$$(\alpha, \beta) \preceq (\alpha', \beta') \Leftarrow \alpha \leq \alpha' \;\; and \;\; \beta \leq \beta'$$

and $X = \{\alpha, \beta, \gamma, \eta, \delta, \lambda\}$ with $\alpha, \beta, \gamma, \eta, \delta, \lambda \in \mathbb{R}^+$,
we define

$$d : X \times X \to \mathbb{A}$$

by

$$d(\alpha,\alpha) = d(\beta,\beta) = d(\gamma,\gamma) = d(\eta,\eta) = d(\delta,\delta) = d(\lambda,\lambda) = (0,0)$$
$$d(\alpha,\beta) = d(\beta,\alpha) = d(\alpha,\gamma) = d(\gamma,\alpha) = d(\alpha,\eta) = d(\eta,\alpha) = d(\alpha,\delta)$$
$$= d(\delta,\alpha) = d(\alpha,\lambda) = d(\lambda,\alpha) = (4,4)$$
$$d(\beta,\gamma) = d(\gamma,\beta) = d(\beta,\eta) = d(\eta,\beta) = d(\beta,\delta) = d(\delta,\beta) = d(\beta,\lambda) = (5,5)$$
$$d(\gamma,\eta) = d(\eta,\gamma) = d(\gamma,\delta) = d(\delta,\gamma) = d(\delta,\lambda) = d(\lambda,\delta) = (1,1)$$
$$d(\eta,\delta) = d(\delta,\eta) = d(\eta,\lambda) = d(\lambda,\eta) = (2,2)$$
$$d(\delta,\lambda) = (6,6) \;\; and \;\; d(\lambda,\delta) = (7,7).$$

It is easy to verify that d is a C^*-algebra valued hexagonal $b-$asymmetric metric space.

Definition 3.3. Let X be a non empty set and $E : X \times X \to \mathbb{A}_I^{'}$.

Suppose the mapping $d : X \times X \to \mathbb{A}$ if it satisfies:

(1) $d(x,y) \succeq \theta$ and $d(x,y) = \theta \Leftrightarrow x = y \; \forall x,y \in X$.
(2) $d(x,y) \preceq E(x,y)[d(x,u) + d(u,v) + d(v,w) + d(w,z) + d(z,y)] \; \forall x,y,u,v,w,z \in X$ and $x \neq u, u \neq v, v \neq w, w \neq z, z \neq y$.

$(X,\mathbb{A},d)$ is called a C^*-algebra valued extended hexagonal $b-$ asymmetric metric space.

Example 3.4. In the example 3.2 we consider $E : X \times X \to \mathbb{A}_I^{'}$ defined by

$$E(x,y) = (x+y, x+y), \quad \forall x,y \in X$$

we have that $(X,\mathbb{A},d)$ is a C^*-algebra valued extended hexagonal $b-$ asymmetric metric space.

Definition 3.5. Let $(X,\mathbb{A},d)$ is a C^*-algebra valued extended hexagonal $b-$ asymmetric metric space. A sequence $\{x_n\}$ in X is said to be:

(i) $\{x_n\}$ $b-$forward (respectively $b-$ backward) converges to $x \in X$ with respect to $\mathbb{A}$ if for all $\varepsilon \succ \theta$, $\exists N_\varepsilon \in \mathbb{N}$ such that

$$d(x,x_n) \preceq \varepsilon, \quad (respectively \;\; d(x_n,x) \preceq \varepsilon).$$

(ii) $\{x_n\}$ converges to x if $lim_{n\to\infty} d(x,x_n) = lim_{n\to\infty} d(x_n,x) = \theta$.
(iii) $\{x_n\}$ is $b-$forward Cauchy sequence respect with $\mathbb{A}$ if $\forall \varepsilon \succ \theta$, $\exists N_\varepsilon \in \mathbb{N}$ such that $d(x_m,x_n) \preceq \varepsilon \;\; \forall m > n \geq N_\varepsilon$.
(iv) $\{x_n\}$ is $b-$backward Cauchy sequence respect with $\mathbb{A}$ if $\forall \varepsilon \succ \theta$, $\exists N_\varepsilon \in \mathbb{N}$ such that $d(x_m,x_n) \preceq \varepsilon \;\; \forall n > m \geq N_\varepsilon$.

Definition 3.6. Let $(X,\mathbb{A},d)$ is a C^*-algebra valued extended hexagonal $b-$ asymmetric metric space. X is said to be $b-$forward (respectively $b-$ backward) complete if every $b-$forward (respectively $b-$ backward) Cauchy sequence $\{x_n\}$ in X, converges to $x \in X$.

Definition 3.7. Let $(X,\mathbb{A},d)$ is a C^*-algebra valued extended hexagonal $b-$ asymmetric metric space. X is said to be complete if X is $b-$forward and $b-$ backward complete.

Lemma 3.8. *Let $(X, \mathbb{A}, d)$ a C^*-algebra valued extended hexagonal $b-$ asymmetric metric space. and $\{x_n\}_n$ be a forward (or backward) Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ forward converges to $x \in X$ and backward converges to $y \in X$, then $x = y$.*

Proof. Let $\varepsilon \succ \theta$. First assume that $\{x_n\}_n$ is a forward Cauchy sequence, so there exists $n_0 \in \mathbb{N}$ such that $\|d(x_n, x_m)\| \leq \frac{\varepsilon}{5\|E(x,y)\|}$ for all $m \geq n \geq n_0$. Since $\{x_n\}_n$forward converges to x so there exists $n_1 \in \mathbb{N}$ such that $\|d(x_n, x)\| \leq \frac{\varepsilon}{5\|E(x,y)\|}$ for all $n \geq n_1$. Also $\{x_n\}_n$ forward converges to y so there exists $n_2 \in \mathbb{N}$ such that$\|d(y, x_n)\| \leq \frac{\varepsilon}{5\|E(x,y)\|}$ for all $n \geq n_2$. Then for all $N \geq \max\{n_0, n_1, n_2\}$,

$$d(x, y) \preceq E(x, y)[d(x, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, y)]$$

$$\begin{aligned}\Rightarrow d(x, y) \leq \|E(x, y)\| \|[d(x, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\ + d(x_{n+3}, y)]\| \leq 5\|E(x, y)\| \frac{\varepsilon}{5\|E(x, y)\|} = \varepsilon.\end{aligned}$$

As $\varepsilon \succ \theta$ was arbitrary, we deduce that $d(x, y) = \theta$, which implies $x = y$.

When $\{x_n\}_n$ is a backward Cauchy sequence, the proof is similar to an earlier state. □

Theorem 3.9. *Let $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued hexagonal $b-$ asymmetric metric space and suppose $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda \ ; \ \forall x, y \in X$$

with $\lambda \in \mathbb{A}$ and $\|\lambda\| < 1$.

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \ \ \forall n \in \mathbb{N}$$

$$\begin{aligned}d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\preceq \lambda^* d(x_n, x_{n-1})\lambda \\ &\preceq (\lambda^*)^2 d(x_{n-1}, x_{n-2})\lambda^2 \\ &. \\ &. \\ &. \\ &\preceq (\lambda^*)^n d(x_1, x_0)\lambda^n,\end{aligned}$$

then $d(x_{n+1}, x_n) \to \theta$ as $n \to \infty$.

For $m \geq 1$ and $r \geq 1$, it follows that

$$\begin{aligned}
d(x_{m+r}, x_m) &\preceq b[d(x_{m+r}, x_{m+r-1}) + d(x_{m+r-1}, x_{m+r-2}) + d(x_{m+r-2}, x_{m+r-3}) + \\
&d(x_{m+r-3}, x_{m+r-4}) + d(x_{m+r-4}, x_m)] \\
&\preceq b[d(x_{m+r}, x_{m+r-1}) + d(x_{m+r-1}, x_{m+r-2}) + d(x_{m+r-2}, x_{m+r-3}) + \\
&d(x_{m+r-3}, x_{m+r-4})] + \\
&b^2[d(x_{m+r-4}, x_{m+r-5}) + d(x_{m+r-5}, x_{m+r-6}) + d(x_{m+r-6}, x_{m+r-7}) + \\
&d(x_{m+r-7}, x_{m+r-8})] + \\
&... + b^{r-1}[d(x_{m+5}, x_{m+4}) + d(x_{m+4}, x_{m+3}) + \\
&d(x_{m+3}, x_{m+2}) + d(x_{m+2}, x_{m+1}) + d(x_{m+1}, x_m)] \\
&\preceq b\sum_{k=1}^{4}(\lambda^*)^{m+r-k} d(x_1, x_0)\lambda^{m+r-k} + ... + b^{r-1}\sum_{k=1}^{4}(\lambda^*)^{m+k} d(x_1, x_0)\lambda^{m+k} + \\
&b^{r-1}(\lambda^*)^m d(x_1, x_0)\lambda^m \\
&\preceq (\|b\|\sum_{k=1}^{4}\|\lambda\|^{2(m+r-k)} + \|d(x_1, x_0)\| + ... + \|b\|^{r-1}\sum_{k=1}^{4}\|\lambda\|^{2(m+k)})\|d(x_1, x_0) + \\
&\|b^{r-1}\|\|\lambda\|^{2m}\|d(x_1, x_0)\|)I \to \theta \;\; as \;\; m \to \infty.
\end{aligned}$$

Similary we obtain $d(x_m, x_{m+r}) \to \theta \;\; as \;\; m \to \infty$.

Consequently, $\{x_n\}$ is $b-$forward and $b-$ backward Cauchy sequence. By completeness of X, there exists $z \in X$ such that $lim_{n\to\infty} x_n = z$.

Now we show that $d(z, Tz) = d(Tz, z) = \theta$.

$$\begin{aligned}
d(Tz, z) &\preceq b[d(Tz, Tx_n) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, z)] \\
&\preceq b[\lambda^* d(z, x_n)\lambda + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, z)] \\
&\Leftrightarrow \|d(z, Tz)\| \leq \|b\|[\|\lambda\|^2\|d(z, x_n)\| + \|d(x_{n+1}, x_{n+2})\| + \|d(x_{n+2}, x_{n+3})\| + \\
&\|d(x_{n+3}, x_{n+4})\| + \|d(x_{n+4}, z)\|] \to \theta \; (n \to \infty).
\end{aligned}$$

Hence $Tz = z$ i.e, z is a fixed point of T.

Unicity:

Let $z' \neq z$ be another fixed point of T.

We have

$$\begin{aligned}
0 \leq \|d(z, z')\| &\leq \|\lambda^* d(z, z')\lambda\| \\
&\leq \|\lambda\|^2\|d(z, z')\|.
\end{aligned}$$

Which is a contradiction ($\|\lambda\|^2 \geq 1$), hence the fixed point z is unique. □

Example 3.10. Let $X = [0, 4]$ and $\mathbb{A} = \mathbb{R}^2$.

Define $d : X \times X \to \mathbb{R}^2$ by

$$d(x,y) = \begin{cases} (|x-y|^6, 0) & \text{if } x \geq y \\ (0, |x-y|^6) & \text{if } x < y. \end{cases}$$

It is easy to verify that $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued hexagonal $b-$ asymmetric metric space. and $Tx = \dfrac{x}{3}$, we have

$$d(Tx,Ty) = \begin{cases} ((\dfrac{|x-y|}{3})^6, 0) & \text{if } x \geq y \\ (0, (\dfrac{|x-y|}{3})^6) & \text{if } x < y. \end{cases}$$

Then $d(Tx,Ty) \preceq \frac{1}{3}I_{\mathbb{R}^2}d(x,y)\frac{1}{3}I_{\mathbb{R}^2}$ where $\|\frac{1}{3}I_{\mathbb{R}^2}\| < 1$, the conditions of Theorem 3.9 are fullfilled. T has a unique fixed point $x = 0$.

Theorem 3.11. *Let $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued hexagonal $b-$ asymmetric metric space and suppose $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \preceq \lambda[d(x,Tx) + d(y,Ty)] \ ; \ \forall x, y \in X$$

with $\lambda \in \mathbb{A}$ and $\|\lambda\| < \frac{1}{2}$.

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \quad \forall n \in \mathbb{N}$$

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\preceq \lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\Rightarrow (I-\lambda)d(x_n, x_{n+1}) \preceq \lambda d(x_{n-1}, x_n) \\ &. \\ &. \\ &. \\ &\preceq \beta^n d(x_0, x_1), \end{aligned}$$

Let $\beta = (I-\lambda)^{-1}(\lambda)$,

since $\|\lambda\| < \frac{1}{2}$ we have $\|\beta\| < 1$.

Then

$$\begin{aligned} (I-\lambda)d(x_n, x_{n+1}) &\preceq \lambda d(x_{n-1}, x_n) \\ &. \\ &. \\ &. \\ &\preceq (\beta)^n d(x_0, x_1), \end{aligned}$$

then $d(x_n, x_{n+1}) \to \theta$ as $n \to \infty$.

For $m \geq 1$ and $r \geq 1$, it follows that

$$\begin{aligned}
d(x_m, x_{m+r}) \preceq\ & b[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+3}) + \\
& d(x_{m+3}, x_{m+4}) + d(x_{m+4}, x_m)] \\
& \preceq b[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+3}) + \\
& d(x_{m+3}, x_{m+4})] \\
& b^2[d(x_{m+4}, x_{m+5}) + d(x_{m+5}, x_{m+6}) + d(x_{m+6}, x_{m+7}) + \\
& d(x_{m+7}, x_{m+8})] + \\
& ... + b^{r-1}[d(x_{m+r-5}, x_{m+r-4}) + d(x_{m+r-4}, x_{m+r-3}) + \\
& d(x_{m+r-3}, x_{m+r-2}) + d(x_{m+r-2}, x_{m+r-1}) + d(x_{m+r-1}, x_{m+r})] \\
& \preceq b \sum_{k=1}^{4} (\beta)^{m+r-k} d(x_0, x_1) + ... + b^{r-1} \sum_{k=1}^{4} (\beta)^{m+k} d(x_0, x_1) + \\
& b^{r-1} \beta^m d(x_0, x_1) \\
& \preceq (\|b\| \sum_{k=1}^{4} \|\beta\|^{2(m+r-k)} + \|d(x_0, x_1)\| + ... + \|b\|^{r-1} \sum_{k=1}^{4} \|\beta\|^{2(m+k)}) \|d(x_0, x_1) + \\
& \|b^{r-1}\| \|\beta\|^{2m} \|d(x_0, x_1)\|) I \to \theta \ \ as \ \ m \to \infty.
\end{aligned}$$

Similary we obtain $d(x_{m+r}, x_m) \to \theta \ \ as \ \ m \to \infty$.

Consequently, $\{x_n\}$ is $b-$forward and $b-$ backward Cauchy sequence. By completeness of X, there exists $z \in X$ such that $lim_{n \to \infty} x_n = z$.

Now we show that $d(z, Tz) = d(Tz, z) = \theta$.

$$\begin{aligned}
d(Tz, z) &\preceq b[d(Tz, Tx_n) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, z)] \\
&\preceq b[\lambda(d(z, Tz) + d(x_n, Tx_n)) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\
&+ d(x_{n+4}, z)] \\
&\Leftrightarrow \|d(z, Tz)\| \leq \|b\| [\|\lambda\| \, \|d(z, x_n)\| + \|\lambda\| \, \|d(x_n, x_{n+1}) + \|d(x_{n+1}, x_{n+2})\| \\
&+ \|d(x_{n+2}, x_{n+3})\| + \|d(x_{n+3}, x_{n+4})\| + \|d(x_{n+4}, z)\|] \to \theta \ (n \to \infty).
\end{aligned}$$

Hence $Tz = z$ i.e, z is a fixed point of T.

Unicity:

Let $z' \neq z$ be another fixed point of T.

We have

$$\begin{aligned}
d(z, z') &\leq \lambda(d(z, Tz) + d(z', Tz')) \\
&= \lambda(d(z, z) + d(z', z')) = \theta
\end{aligned}$$

which is a contradiction ($d(z, z') = \theta \Rightarrow z = z'$), hence the fixed point z is unique. □

Theorem 3.12. *Let $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued extended hexagonal $b-$ asymmetric metric space and suppose $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \preceq \lambda^* E(x, y) d(x, y) \lambda \ ; \ \forall x, y \in X$$

with $\lambda \in \mathbb{A}$, $\|\lambda\| < 1$ *and* $sup_{m\geq 1} lim_{n\to\infty}\|E(x_{n+1}, x_n)\|\|E(x_n, x_m)\| < \dfrac{1}{\|\lambda\|^8}$,

and $lim_{n,m\to\infty}\|E(x_n, x_m)\| < \dfrac{1}{\|\lambda\|^2}$.

Then T *has a unique fixed point in* X.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \quad \forall n \in \mathbb{N}$$

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\preceq \lambda^* E(x_n, x_{n-1}) d(x_n, x_{n-1})\lambda \\ &\preceq (\lambda^*)^2 E(x_n, x_{n-1}) E(x_{n-1}, x_{n-2}) d(x_{n-1}, x_{n-2})\lambda^2 \\ &. \\ &. \\ &. \\ &\preceq (\lambda^*)^n \prod_{k=1}^{n} E(x_k, x_{k-1}) d(x_1, x_0)\lambda^n \end{aligned}$$

then $d(x_{n+1}, x_n) \to \theta$ as $n \to \infty$.

For $m \geq 1$ and $r \geq 1$, it follows that

$$\begin{aligned} d(x_{m+r}, x_m) \preceq\ & E(x_{m+r}, x_m)[d(x_{m+r}, x_{m+r-1}) + d(x_{m+r-1}, x_{m+r-2}) + d(x_{m+r-2}, x_{m+r-3}) + \\ & d(x_{m+r-3}, x_{m+r-4}) + d(x_{m+r-4}, x_m)] \\ & \preceq E(x_{m+r}, x_m)[d(x_{m+r}, x_{m+r-1}) + d(x_{m+r-1}, x_{m+r-2}) + d(x_{m+r-2}, x_{m+r-3}) + \\ & d(x_{m+r-3}, x_{m+r-4})] + \\ & E(x_{m+r}, x_m)E(x_{m+r-4}, x_m)[d(x_{m+r-4}, x_{m+r-5}) + d(x_{m+r-5}, x_{m+r-6}) \\ & + d(x_{m+r-6}, x_{m+r-7}) + d(x_{m+r-7}, x_{m+r-8})] + \\ & ... + E(x_{m+r}, x_m)E(x_{m+r-4}, x_m)...E(x_{m+1}, x_m)[d(x_{m+5}, x_{m+4}) + d(x_{m+4}, x_{m+3}) \\ & + d(x_{m+3}, x_{m+2}) + d(x_{m+2}, x_{m+1}) + d(x_{m+1}, x_m)] \\ & = \sum_{k=m}^{m+r} \prod_{j=m+1}^{k} E(x_j, x_m))[d(x_{k+4}, x_{k+3}) + d(x_{k+3}, x_{k+2}) \\ & + d(x_{k+2}, x_{k+1}) + d(x_{k+1}, x_k)] + \prod_{j=m+1}^{m+r} E(x_j, x_m))d(x_{m+1}, x_m) \\ & \preceq \sum_{k=m}^{m+r} \prod_{j=m+1}^{k} E(x_j, x_m))[(\lambda^*)^{k+3} \prod_{k=1}^{n} E(x_k, x_{k-1}) d(x_1, x_0)\lambda^{k+3} \\ & + (\lambda^*)^{k+2} \prod_{k=1}^{n} E(x_k, x_{k-1}) d(x_1, x_0)\lambda^{k+2} + (\lambda^*)^{k+1} \prod_{k=1}^{n} E(x_k, x_{k-1}) d(x_1, x_0)\lambda^{k+1} \end{aligned}$$

$$+ (\lambda^*)^k \prod_{k=1}^{n} E(x_k, x_{k-1}) d(x_1, x_0)\lambda^k] + \prod_{j=m+1}^{m+r} E(x_j, x_m))(\lambda^*)^m \prod_{k=1}^{n} E(x_k, x_{k-1}) d(x_1, x_0)\lambda^m$$

$$\preceq \|d(x_1, x_0)\| [\sum_{k=m}^{m+r} \prod_{j=m+1}^{k} \|E(x_j, x_m)\| \| \prod_{k=1}^{n} E(x_k, x_{k-1})\| \|\lambda\|^{2(k+3)}$$

$$+ \|\lambda\|^{2(k+2)} \| \prod_{k=1}^{n} E(x_k, x_{k-1})\| + \|\lambda^*\|^{2(k+1)} \| \prod_{k=1}^{n} E(x_k, x_{k-1})\| +$$

$$\|\lambda^*\|^{2k} \| \prod_{k=1}^{n} E(x_k, x_{k-1})\|]I + \|d(x_1, x_0)\| \| \prod_{j=m+1}^{m+r} E(x_j, x_m))\| \|\lambda\|^{2m} \| \prod_{k=1}^{n} E(x_k, x_{k-1})\|]I$$

$$\to \theta \ \ as \ m \to \infty.$$

Similary we obtain $d(x_m, x_{m+r}) \to \theta \ \ as \ \ m \to \infty$.

Consequently, $\{x_n\}$ is $b-$forward and $b-$ backward Cauchy sequence. By completeness of X, there exists $z \in X$ such that $lim_{n\to\infty} x_n = z$.

Now we show that $d(z, Tz) = d(Tz, z) = \theta$

$$\begin{aligned}
&d(Tz, z)\\
&\preceq E(Tz, z)[d(Tz, Tx_n) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, z)]\\
&\preceq E(Tz, z)[\lambda^* d(z, x_n)\lambda + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, z)]\\
&\Leftrightarrow \|d(z, Tz)\| \leq \|E(Tz, z)\| [\|\lambda\|^2 \|d(z, x_n)\| + \|d(x_{n+1}, x_{n+2})\| + \|d(x_{n+2}, x_{n+3})\| +\\
&\|d(x_{n+3}, x_{n+4})\| + \|d(x_{n+4}, z)\|] \to \theta \ (n \to \infty).
\end{aligned}$$

Hence $Tz = z$ i.e, z is a fixed point of T.

Unicity:

Let $z' \neq z$ be another fixed point of T.

We have

$$\begin{aligned}
0 \leq \|d(z, z')\| &\leq \|\lambda^* E(z, z') d(z, z')\lambda\|\\
&\leq \|\lambda\|^2 \|E(z, z')\| \|d(z, z')\|\\
&< \|\lambda\|^2 \frac{1}{\|\lambda\|^2} \|d(z, z')\|\\
&< \|d(z, z')\|,
\end{aligned}$$

which is a contradiction, hence the fixed point z is unique. □

Example 3.13. Consider $T : X \to X$ defined by $Tx = \dfrac{x}{3}$ in the Example 3.10 and $E : X \times X \to \mathbb{A}'_I$ defined by

$$E(x, y) = (x + y, x + y) \ \ \forall x, y \in X,$$

we have

$$d(Tx, Ty) = \begin{cases} ((\frac{|x-y|}{3})^6, 0) & \text{if } x \geq y. \\ (0, (\frac{|x-y|}{3})^6) & \text{if } x < y. \end{cases}$$

With
$d(Tx,Ty) \preceq \frac{1}{3}I_{\mathbb{R}^2}d(x,y)\frac{1}{3}I_{\mathbb{R}^2}$, $\|\frac{1}{4}I_{\mathbb{R}^2}\| < 1$ and $4 > 1$
and
$d(Tx,Ty) \preceq \frac{1}{3}I_{\mathbb{R}^2}(x+y,x+y)d(x,y)\frac{1}{3}I_{\mathbb{R}^2}$
Then T has a unique fixed point.

Theorem 3.14. *Let $(X,\mathbb{A},d)$ is a complete C^*-algebra valued extended hexagonal $b-$ asymmetric metric space and suppose $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \preceq \lambda^* k d(x,y)\lambda \ ; \ \forall x,y \in X$$

with $\lambda \in \mathbb{A}$, $k \in \mathbb{A}'_I$ and $\|\lambda\| < 1, \quad \|k\| > 1.$

Then T has a unique fixed point in X.

Proof. If we put $E(x,y) = k$ in Theorem 3.12 we obtain the result. □

Example 3.15. Let $\mathbb{A} = M_2(\mathbb{R})$ of all 2×2 matrices with the usual addition ,scalar multiplication and multiplication.

Define partial ordering on $\mathbb{A}$ as $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \Leftrightarrow a_i \geq b_i$
for $i = 1,2,3,4$.

For any $A \in \mathbb{A}$ we define its norm as, $\| \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \| = \left[\sum_{i=1}^{i=4} |a_i|^2\right]^{\frac{1}{2}}$.

Let $X = A \cup B$, where $A = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}$ and $B = [1,2]$.
Define $d : X \times X \to [0,+\infty[$ as follows:

$$\begin{cases} d(x,y) = d(y,x) \ for \ all \ x,y \in B. \\ d(x,y) = 0 \Leftrightarrow y = x \ for \ all \ x,y \in X. \end{cases}$$

and

$$\begin{cases} d\left(\frac{1}{3},\frac{1}{4}\right) = d\left(0,\frac{1}{2}\right) = d(\frac{1}{4},\frac{1}{6}) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \\ d\left(\frac{1}{3},0\right) = d\left(\frac{1}{4},\frac{1}{2}\right) = d(\frac{1}{5},\frac{1}{2}) = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.4 \end{pmatrix} \\ d\left(0,\frac{1}{3}\right) = d\left(\frac{1}{2},\frac{1}{4}\right) = \begin{pmatrix} 0.45 & 0 \\ 0 & 0.45 \end{pmatrix} \\ d\left(\frac{1}{3},\frac{1}{2}\right) = d\left(\frac{1}{3},\frac{1}{2}\right) = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix} \\ d\left(x,y\right) = \begin{pmatrix} |x-y| & 0 \\ 0 & |x-y| \end{pmatrix} \ otherwise. \end{cases}$$

Then $(X,\mathbb{A}_+,d)$ is a C^*-algebra valued extended hexagonal $b-$ asymmetric metric space with $E(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\|E(x,y)\| \geq 1$.

Define mapping $T : X \to X$ by

$$T(x) = \begin{cases} x^{\frac{1}{2}} \; if \; x \in [1,2] \\ 1 \; if \; x \in A. \end{cases}$$

Evidently, $T(x) \in X$. Consider the following possibilities:
case $1 : x, y \in [1,2] \;\; x \neq y$. Then

$$T(x) = x^{\frac{1}{2}}, \; T(y) = y^{\frac{1}{2}}, \; d\left(Tx, Ty\right) = \begin{pmatrix} x^{\frac{1}{2}} - y^{\frac{1}{2}} & 0 \\ 0 & x^{\frac{1}{2}} - y^{\frac{1}{2}} \end{pmatrix}.$$

On the other hand

$$d\left(x, y\right) = \begin{pmatrix} x - y & 0 \\ 0 & x - y \end{pmatrix}.$$

it follows that

$$d(Tx, Ty) \preceq \lambda^{*} E(x, y) d(x, y) \lambda.$$

Indeed

$$\begin{aligned} d(Tx, Ty) &= \begin{pmatrix} x^{\frac{1}{2}} - y^{\frac{1}{2}} & 0 \\ 0 & x^{\frac{1}{2}} - y^{\frac{1}{2}} \end{pmatrix} \\ &\preceq \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x - y & 0 \\ 0 & x - y \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \\ &= \lambda^{*} E(x, y) d(x, y) \lambda. \end{aligned}$$

where

$$\lambda = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

with verify

$$\|\lambda\| = \frac{\sqrt{2}}{\sqrt{5}} \leq 1.$$

case $2 : x \in [1,2], y \in A$. Then

$$T(x) = x^{\frac{1}{2}}, \; T(y) = 1$$

$$d\left(Tx, Ty\right) = \begin{pmatrix} x^{\frac{1}{2}} - 1 & 0 \\ 0 & x^{\frac{1}{2}} - 1 \end{pmatrix}.$$

On the other hand

$$d\left(x, y\right) = \begin{pmatrix} x - y & 0 \\ 0 & x - y \end{pmatrix}.$$

It follows that

$$d(Tx, Ty) \preceq \lambda^{*} E(x, y) d(x, y) \lambda.$$

Indeed

$$\begin{aligned} d(Tx,Ty) &= \begin{pmatrix} x^{\frac{1}{2}}-1 & 0 \\ 0 & x^{\frac{1}{2}}-1 \end{pmatrix} \\ &\preceq \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x-1 & 0 \\ 0 & x-1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \\ &\preceq \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x-y & 0 \\ 0 & x-y \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \\ &= \lambda^* E(x,y) d(x,y) \lambda. \end{aligned}$$

Where

$$\lambda = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

with verify

$$\|\lambda\| = \frac{\sqrt{2}}{\sqrt{5}} \leq 1.$$

Therefore, T has a unique fixed point $z = 1$.

Example 3.16. Let $X = \mathbb{R}_+$ and $\mathbb{A} = M_2(\mathbb{R}_+)$ of all 2×2 matrices with the usual addition ,scalar multiplication and multiplication. Define partial ordering on $\mathbb{A}$ as $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \Leftrightarrow a_i \geq b_i$ for $i = 1, 2, 3, 4$.

For any $A \in \mathbb{A}$ we define its norm as, $\| \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \| = \left[\sum_{i=1}^{i=4} |a_i|^2 \right]^{\frac{1}{2}}$.
Define $d : X \times X \to M_2(\mathbb{R}_+)$ as follows:

$$\begin{cases} d(x,y) = \begin{pmatrix} e^x - e^y & 0 \\ 0 & 0 \end{pmatrix} & if\ x \geq y \\ d(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & e^{-x} - e^{-y} \end{pmatrix} & if\ x \leq y \end{cases}$$

and $E : X \times X \to M_2(\mathbb{R}_+),\ E(x,y) = \begin{pmatrix} e^x + e^y & 0 \\ 0 & 0 \end{pmatrix}$.

Then $(X, \mathbb{A}_+, d)$ is a C^*-algebra valued extended hexagonal $b-$ asymmetric metric space.

Define mapping $T : X \to X$ by

$$T(x) = \frac{x}{3}.$$

Evidently, $T(x) \in X$. Then

$$\begin{cases} d\left(Tx,Ty\right) = \begin{pmatrix} e^{\frac{x}{3}} - e^{\frac{y}{3}} & 0 \\ 0 & 0 \end{pmatrix} \; if \; x \geq y. \\ d\left(Tx,Ty\right) = \begin{pmatrix} 0 & 0 \\ 0 & e^{-\frac{x}{3}} - e^{-\frac{y}{3}} \end{pmatrix} \; if \; x \leq y. \end{cases}$$

It follows that

$$d(Tx,Ty) \preceq \lambda^* E(x,y) d(x,y) \lambda.$$

Indeed

$$\begin{cases} d\left(Tx,Ty\right) \preceq \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} e^{x} + e^{y} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{x} - e^{y} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \; if \; x \geq y. \\ d\left(Tx,Ty\right) \preceq \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} e^{x} + e^{y} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e^{-x} - e^{-y} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \; if \; x \leq y. \end{cases}$$

Where

$$\lambda = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

with verify

$$\|\lambda\| = \frac{1}{2} < 1.$$

Therefore, T has a unique fixed point $z = 0$.

Definition 3.17. Let X be a nonempty set, a mapping T is C^*-algebra valued extended hexagonal $b-$ asymmetric expansion mapping on X, if $T : X \to X$ satisfies:

(1) $T(X) = X$.

(2) $d(Tx,Ty) \succeq \lambda^* E(x,y) d(x,y) \lambda$ with $\lambda \in \mathbb{A}$ and $\|\lambda^{-1}\| < 1$.

Theorem 3.18. *Let $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued extended hexagonal $b-$asymmetric metric space and suppose $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \succeq \lambda^* E(x,y) d(x,y) \lambda \;\; \forall x,y \in X,$$

with $\lambda \in \mathbb{A}$, $\|\lambda\| < 1$.

Then T has a unique fixed point in X.

Proof. Firstly T is injective. We have for any $x, y \in X$ with $x \neq y$ if $Tx = Ty$.

$$\theta = d(Tx,Ty) \succeq \lambda^* E(x,y) d(x,y) \lambda$$
$$\Rightarrow d(x,y) = \theta.$$

Which is a contradiction. Thus T is injective.

Substitute x, y with $T^{-1}x, T^{-1}y$, respectively, in

$$d(Tx,Ty) \succeq \lambda^* E(x,y) d(x,y) \lambda$$

and we get

$$d(x,y) \succeq \lambda^* E(T^{-1}x, T^{-1}y) d(T^{-1}x, T^{-1}y) \lambda.$$

Then

$$(\lambda^{-1})^* d(x,y)\lambda^{-1} \succeq E(T^{-1}x, T^{-1}y)d(T^{-1}x, T^{-1}y) \succeq d(T^{-1}x, T^{-1}y).$$

Using Theorem 3.9, there exists a unique x such $T^{-1}x = x$, which means there has a unique fixed point $x \in X$ such that $Tx = x$. □

4. Application

As application of theorem on complete C^*-algebra valued extended hexagonal $b-$asymmetric metric spaces, existence and uniqueness results for a type of following integral equation

$$x(t) = \int_U K(t,s,x(t))ds + f(t) \ t \in U,$$

where U is a Lebesgue measurable set.

Suppose that

(1) $K : U \times U \times \mathbb{R} \to \mathbb{R}$ and $f \in L^\infty(U)$.
(2) There exists a continuous function $\varphi : U \times U \to \mathbb{R}$ and $\alpha \in (0,1)$ such that

$$|K(t,s,u) - K(t,s,v)| \leq \alpha|\varphi(t,s)(u-v)| \ \forall t,s \in U \ \ and \ u,v \in \mathbb{R}.$$

(3) $sup_{t\in U} \int_U |\varphi(t,s)|ds \leq 1$.

Then, the integral equation has a unique solution $z \in L^\infty(U)$.

Proof. Let $X = L^\infty(U)$, $H = L^2(U)$ and $\mathbb{A} = B(H)$.

Define a C^*-algebra valued extended hexagonal $b-$ asymmetric metric

$$d : X \times X \to \mathbb{A}$$

by

$$\begin{cases} d\,(x,y) = \pi_{x-y} \ if \ x \geq y. \\ d\,(f,g) = \pi_{y-x} \ if \ y \leq x. \end{cases}$$

Where $\pi_h(\varphi) : H \to H$ is the multiplication operator defined by $\pi_h(\varphi) = h.\varphi$ for $\varphi \in H$.

Then d is a C^*-algebra valued extended hexagonal $b-$ asymmetric metric.

Define $T : X \to X$ by

$$Tx(t) = \int_U K(t,s,x(t))ds + f(t), \ t \in U.$$

If $x \geq y$ we have, for any $h \in H$

$$\begin{aligned}
\|d(Tx,Ty)\| &= sup_{\|h\|=1}(\pi_{x-y}h,h) \\
&= sup_{\|h\|=1}\int_U [|\int_U K(t,s,x(t)) - K(t,s,y(t))ds|]h(t)\overline{h(t)}dt \\
&\leq sup_{\|h\|=1}\int_U [\int_U |K(t,s,x(t)) - K(t,s,y(t))|ds]|h(t)|^2 dt \\
&\leq sup_{\|h\|=1}\int_U [\int_U |\alpha\varphi(t,s)(x(s)-y(s))|ds]|h(t)|^2 dt \\
&\leq \alpha sup_{\|h\|=1}\int_U [\int_U |\varphi(t,s)|ds]|h(t)|^2 dt.\|x-y\|_\infty \\
&\leq \alpha sup_{\|h\|=1}\int_U [|\varphi(t,s)|ds].sup_{\|h\|=1}\int_U |h(t)|^2 dt.\|x-y\|_\infty \\
&\leq \alpha\|x-y\|_\infty \\
&= \alpha\|d(x,y)\|.
\end{aligned}$$

In the same way for $x \geq y$.

Then the integral equation has a unique solution $z \in L^\infty(U)$. □

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