Functional Analysis FAC – SEAMS
Volume 2, 2023, ID 2

Some Fixed Point Theorems in Rectangular b-Metric Spaces

Mohamed Rossafi [1] ✉, Hafida Massit [2]

[1] *LaSMA Laboratory Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, B. P. 1796 Fes Atlas, Morocco*

[2] *Department of Mathematics, University of Ibn Tofail, B.P. 133, Kenitra, Morocco*

ABSTRACT. This present paper extends some fixed point theorems for a class of nonexpansive mappings using Picard sequences in rectangular b-metric spaces endowed with an arbitrary binary relation. Our results extend and improve many results existing in the literature.

1. Introduction and Preliminaries

In 1922 Banach proved the following theorem.

Theorem 1.1. *[1] Let (X, d) be a complete metric space. Let $T : X \to X$ be a contraction. Then:*

(i) *T has a unique fixed point $x \in X$.*
(ii) *For every $x_0 \in X$, the sequence (x_n), where $x_{n+1} = Tx_n$, converges to x.*
(iii) *We have the following estimate: For every $x \in X$, $d(x_n, x) \leq \dfrac{k^n}{1-k} d(x_0, x_1)$, $n \in \mathbb{N}$.*

Several generalization of the previous theorem in various directions are given [4–12].

In 2000, for the first time generalized metric spaces were introduced by Branciari [2], in such a way that triangle inequality is replaced by the quadrilateral inequality

$$d(x, y) \leq d(x, z) + d(z, u) + d(u, y),$$

for all pairwise distinct points x, y, z and u. Any metric space is a generalized metric space but in general, generalized metric space might not be a metric space.

In 2015 George et al. [3] announced the notion of $b-$rectangular metric space as follow:

Definition 1.2. [3]. Let X be a nonempty set, $s \geq 1$ be a given real number, and let d: $X \times X \to [0, +\infty[$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

1. $d(x, y) = 0$, if only if $x = y$;
2. $d(x, y) = d(y, x)$;

Date submitted: 2022-11-03.
2020 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Key words and phrases. Fixed point, nonexpansive mapping, rectangular b-metric spaces.

3. $d(x,y) \leq s\left[d(x,u)+d(u,v)+d(v,y)\right]$ $(b-rectangular\ inequality)$.

Then (X,d) is called a $b-$rectangular metric space.

Example 1.3. Let $X = A \cup B$ such that $A = \{\frac{1}{n}; n \in \mathbb{N}^*\}$ and $B = \{2,3\}$. Define $d : X \times X \to [0, \infty[$ by

$$d(x,y) = \begin{cases} 2\sigma & , x, y \in \ A \\ \frac{\sigma}{2n} & , x \in \ A\, and\, y \in B \ \ or \ \ x \in \ \ B\, and\, y \in A \\ 0 & , x = y \\ \sigma \ \ otherwise & \end{cases}$$

where $\sigma > 0$. d is a rectangular $b-$ metric with a coefficient $b = 2$ and (X,d) is a complete rectangular $b-$metric.

Definition 1.4. [3] Let (X,d) be a rectangular $b-$metric space and $\{x_n\}$ a sequence in X. We say that:

1. $\{x_n\}$ converges to $x \in X$, if $d(x_n, x) \to 0$ as $n \to \infty$.
2. $\{x_n\}$ is a Cauchy sequence, if $d(x_m, x_n) \to 0$ as $m, n \to \infty$
3. (X,d) is complete, if very Cauchy sequence in X is convergent.

Let (X,d) be a rectangular $b-$metric space and $T : X \to X$ be a map on X. We denote the set of fixed point of T by $F(T) = \{x \in X : Tx = x\}$.

Let W be a subset of $X \times X$ and let $T : X \to X$ be a map. W is Banach $T-$invariant if $(Tx, T^2x) \in W$ whenever $(x, Tx) \in W$. Also, a subset Y of X is well ordered with respect to W if for all $x, y \in Y$ we have $(x,y) \in W$ or $(y,x) \in W$.

In this paper, we provide some fixed point results for nonexpansive mapping in rectangular $b-$metric spaces endowed with an arbitrary binary relation. Using the Picard sequence. Our main results extends and improves many results existing in the literature.

2. Main results

The following lemmas use to proof ours results:

Lemma 2.1. *Let (X,d) be a rectangular $b-$metric space with $b \geq 1$ and*

$T : X \to X$ be a mapping. Suppose that $\{x_n\}$ a sequence in X defined by $x_{n+1} = Tx_n$ $n \in \mathbb{N}$. such that

$$d(x_n, x_{n+1}) \leq \delta d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$ and $\delta \in [0,1)$.

Then $\{x_n\}$ is a Cauchy sequence.

Proof. Let $x_0 \in X$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$

We have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \delta d(x_{n-1}, x_n) \\ &\leq \delta^2 d(x_{n-2}, x_{n-1}) \\ &\ \ . \end{aligned}$$

.

$$\leq \delta^n d(x_0, x_1).$$

For $m \geq 1$ and $p \geq 1$, it follows that

$d(x_{m+p}, x_m) \leq b[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m)]$

$\leq bd(x_{m+p}, x_{m+p-1}) + bd(x_{m+p-1}, x_{m+p-2}) + b[b[d(x_{m+p-2}, x_{m+p-3}) + d(x_{m+p-3}, x_{m+p-4}) + d(x_{m+p-4}, x_m)]]$

$= bd(x_{m+p}, x_{m+p-1}) + bd(x_{m+p-1}, x_{m+p-2}) + b^2 d(x_{m+p-2}, x_{m+p-3}) + b^2 d(x_{m+p-3}, x_{m+p-4}) + b^2 d(x_{m+p-4}, x_m)$

$\leq bd(x_{m+p}, x_{m+p-1}) + bd(x_{m+p-1}, x_{m+p-2}) + b^2 d(x_{m+p-2}, x_{m+p-3}) + b^2 d(x_{m+p-3}, x_{m+p-4}) + ... + b^{\frac{p-1}{2}} d(x_{m+3}, x_{m+2}) + b^{\frac{p-1}{2}} d(x_{m+2}, x_{m+1}) + b^{\frac{p-1}{2}} d(x_{m+1}, x_m)$

$\leq [b\delta^{m+p-1} + b\delta^{m+p-1} + ... + b^{\frac{p-1}{2}} \delta^m] d(x_0, x_1) \to 0$ as $m \to \infty$.

Wich implies that $\{x_n\}$ is a Cauchy sequence. □

Lemma 2.2. *Let $\{x_n\}$ a nonincreasing sequence of nonnegative real numbers, then for $\alpha < \beta$ and $b \geq 1$, the sequence*

$$\{\frac{(4+4b)x_n + \alpha}{(4+4b)x_n + \beta}\}$$

is nonincreasing too.

Proof. We have

$((4+4b)x_n + \alpha)((4+4b)x_{n+1} + \beta) \geq ((4+4b)x_{n+1} + \alpha)((4+4b)x_n + \beta)$

$\Leftrightarrow \dfrac{(4+4b)x_n + \alpha}{(4+4b)x_n + \beta} \geq \dfrac{(4+4b)x_{n+1} + \alpha}{(4+4b)x_{n+1} + \beta}.$

Since $\{x_n\}$ is nonincreasing sequence, we obtain the result. □

Corollary 2.3. *Let (X, d) be a complete rectangular $b-$metric space with coefficient $b \geq 1$, $T : X \to X$ be a nonexpansive mapping and $x_0 \in X$.*

If $\{x_n\}$ is a Picard sequence, then the sequence

$$\{\frac{(4+4b)d(x_n, x_{n+1}) + \alpha}{(4+4b)d(x_n, x_{n+1}) + \beta}\}$$

is nonincreasing for $\alpha < \beta$

Theorem 2.4. *Let (X, d) be a complete rectangular $b-$ metric space with coefficient $b \geq 1$ endowed with a binary relation W on X and let $T : X \to X$ be a nonexpansive mapping such that*

$$bd(Tx, Ty) \leq \left(\frac{d(x,Ty) + d(y,Tx) + d(x,T^2y) + d(y,T^2x) + d(Tx,T^2y) + d(Ty,T^2x) + \alpha}{d(x,Tx) + d(y,Ty) + d(x,T^2x) + d(y,T^2y) + d(Tx,T^2x) + d(Ty,T^2y) + \beta} + \lambda\right) d(x,y) \tag{1}$$

for all $(x, y) \in W$, where $\lambda \in [0, 1)$,$\alpha, \beta \in [0, +\infty)$ such that $\alpha < \beta$

Assume that:

(a) *W is Banach $T-$ invariant.*

(b) *if $\{u_n\}$ is a sequence in X such that $(u_{n-1}, u_n) \in W$ for all $n \in \mathbb{N}$ and $u_n \to u \in X$ as $n \to \infty$, then $(u_{n-1}, u) \in W$,$\forall n \in \mathbb{N}$*

(c) *$F(T)$ is well ordered with respect to W.*

If there exists $u_0 \in X$ such that $(u_0, Tu_0) \in W$ and

$$\frac{(4+4b)d(u_0,Tu_0)+\alpha}{(4+4b)d(u_0,Tu_0)+\beta}+\lambda<1 \ (2)$$

Then,

(i) *T has a fixed point $u \in X$*

(ii) *for any $u_0 \in X$, the sequence $\{u_n\}$ convergs to a fixed point of T*

(iii) *if $u, v \in X$ are two different fixed points of T, then*

$$d(u,v) \geq max\{\frac{\beta(b-\lambda)-\alpha}{6}, 0\}.$$

Proof. Let $\{u_n\}$ be a sequence in X with an initial approximation $u_0 \in X$ such that $Tu_n = u_{n+1}$.

Assume that $(u_0, Tu_0) \in W$, then (2) holds.

If $u_n = u_{n-1}$ for some $n \in \mathbb{N}$, then u_{n-1} is a fixed point of T .

Now we assume that $u_{n-1} \neq u_n \ \forall n \in \mathbb{N}$.

Since W is Banach $T-$ invariant,

we get that $(u_1, u_2) = (Tu_0, T^2u_0) \in W$ for $(u_0, u_1) = (u_0, Tu_0) \in W$.

Using (1) with $x = u_{n-1}$ and $y = u_n \ \forall n \in \mathbb{N}$,

we have

$$bd(u_n,u_{n+1}) = bd(Tu_{n-1},Tu_n)$$
$$\leq \left(\frac{d(u_{n-1},u_{n+1})+d(u_n,u_n)+d(u_{n-1},u_{n+2})+d(u_n,u_{n+1})+d(u_n,u_{n+2})+d(u_{n+1},u_{n+1})+\alpha}{d(u_{n-1},u_n)+d(u_n,u_{n+1})+d(u_{n-1},u_{n+1})+d(u_n,u_{n+2})+d(u_n,u_{n+1})+d(u_{n+1},u_{n+2})+\beta}+\lambda\right)d(u_{n-1},u_n)$$
$$\leq \left(\frac{d(u_{n-1},u_{n+1})+b(d(u_{n-1},u_n)+d(u_n,u_{n+1})+d(u_{n+1},u_{n+2}))+d(u_n,u_{n+1})+d(u_n,u_{n+2})+\alpha}{d(u_{n-1},u_n)+d(u_n,u_{n+1})+d(u_{n-1},u_{n+1})+d(u_n,u_{n+2})+d(u_n,u_{n+1})+d(u_{n+1},u_{n+2})+\beta}+\lambda\right)d(u_{n-1},u_n)$$
$$\leq \left(\frac{d(u_{n-1},u_{n+1})+bd(u_{n-1},u_n)+bd(u_n,u_{n+1})+bd(u_{n+1},u_{n+2})+d(u_n,u_{n+1})+\alpha}{d(u_{n-1},u_n)+d(u_n,u_{n+1})+d(u_{n-1},u_{n+1})+d(u_n,u_{n+2})+d(u_n,u_{n+1})+d(u_{n+1},u_{n+2})+\beta}+\lambda\right)d(u_{n-1},u_n)$$
$$\leq b\left(\frac{d(u_{n-1},u_{n+1})+d(u_{n-1},u_n)+d(u_n,u_{n+1})+d(u_{n+1},u_{n+2})+d(u_n,u_{n+1})+d(u_{n+1},u_{n+2})\alpha}{d(u_{n-1},u_n)+d(u_n,u_{n+1})+d(u_{n-1},u_{n+1})+d(u_n,u_{n+2})+d(u_n,u_{n+1})+d(u_{n+1},u_{n+2})+\beta}+\lambda\right)d(u_{n-1},u_n)$$

Then,

$$d(u_n,u_{n+1}) \leq \left(\frac{(1+b)d(u_{n-1},u_n)+(2b+2)d(u_n,u_{n+1})+(1+b)d(u_{n+1},u_{n+2})+\alpha}{(1+b)d(u_{n-1},u_n)+(2b+2)d(u_n,u_{n+1})+(1+b)d(u_{n+1},u_{n+2})+\beta}+\lambda\right)d(u_{n-1},u_n)$$

Since T is nonexpansive mapping,

we get

$$d(u_n,u_{n+1}) \leq \left(\frac{(1+b)d(u_0,u_1)+(2b+2)d(u_1,u_2)+(1+b)d(u_2,u_3)+\alpha}{(1+b)d(u_0,u_1)+(2b+2)d(u_1,u_2)+(1+b)d(u_2,u_3)+\beta}+\lambda\right)d(u_{n-1},u_n)$$
$$= \left(\frac{(1+b)d(u_0,Tu_0)+(2b+2)d(Tu_0,T^2u_0)+(1+b)d(T^2u_0,T^3u_0)+\alpha}{(1+b)d(u_0,Tu_0)+(2b+2)d(Tu_0,T^2u_0)+(1+b)d(T^2u_0,T^3u_0)+\beta}+\lambda\right)d(u_{n-1},u_n)$$
$$\leq \left(\frac{(4+4b)d(u_0,Tu_0)+\alpha}{(4+4b)d(u_0,Tu_0)+\beta}+\lambda\right)d(u_{n-1},u_n)$$
$$= \delta d(u_{n-1},u_n)$$

Where

$$\delta = \frac{(4+4b)d(u_0,Tu_0)+\alpha}{(4+4b)d(u_0,Tu_0)+\beta}+\lambda<1$$

Applying Lemma 2.5 we have $\{u_n\}$ is Cauchy sequence.

By the completeness of X, there exists $x \in X$ such that $lim_{n\to\infty}u_n = u$.

From hypothesis (b) we have $(u_n, u) \in W$;

taking $x = u_n$ and $y = u$ in (1),
we get
$bd(u_{n+1}, Tu) = bd(Tu_n, Tu)$
$\leq \quad (\frac{d(u_n,Tu)+d(u,Tu_n)+d(u_n,T^2u)+d(u,T^2u_n)+d(Tu_n,T^2u)+d(Tu,T^2u_n)+\alpha}{d(u_n,Tu_n)+d(u,Tu)+d(u_n,T^2u_n)+d(u,T^2u)+d(Tu_n,T^2u_n)+d(Tu,T^2u)+\beta} + \lambda)d(u_n,u)$
$\leq \quad (\frac{d(u_n,Tu)+d(u,u_{n+1})+d(u_n,T^2u)+d(u,u_{n+2})+d(u_{n+1},T^2u)+d(Tu,u_{n+2})+\alpha}{d(u_n,u_{n+1})+d(u,Tu)+d(u_n,u_{n+2})+d(u,T^2u)+d(u_{n+1},u_{n+2})+d(Tu,T^2u)+\beta} + \lambda)d(u_n,u)$ (3)
Taking limit as $n \to \infty$ in (3) ,
we obtain that $d(u, Tu) \leq 0$ then $Tu = u$,
in other words u is fixed point of T.Thus (i) and (ii) hold.
Let v be a another fixed point of T with $u \neq v$.
Using (1) with $x = u$ and $y = v$, we get
$bd(u, v) = bd(Tu, Tv)$
$\leq (\frac{d(u,Tv)+d(v,Tu)+d(u,T^2v)+d(v,T^2u)+d(Tu,T^2v)+d(Tv,T^2u)+\alpha}{d(u,Tu)+d(v,Tv)+d(u,T^2u)+d(v,T^2v)+d(Tu,T^2u)+d(Tv,T^2v)+\beta} + \lambda)d(u,v)$
$= (\frac{6d(u,v)+\alpha}{\beta} + \lambda)d(u,v)$
and hence $d(u, v) \geq \frac{\beta(b-\lambda)-\alpha}{6}$, that is (iii) holds. □

Theorem 2.5. *Let (X, d) be a complete rectangular $b-$ metric space with coefficient $b \geq 1$ endowed with a binary relation W on X and let $T : X \to X$ be a nonexpansive mapping such that*

$$bd(Tx,Ty) \leq (\frac{d(x,Ty)+d(y,Tx)+d(x,T^2y)+d(y,T^2x)+d(Tx,T^2y)+d(Ty,T^2x)+\alpha}{d(x,Tx)+d(y,Ty)+d(x,T^2x)+d(y,T^2y)+d(Tx,T^2x)+d(Ty,T^2y)+\beta} + \lambda)d(x,y) + Ld(y,Tx) \quad (4)$$

for all $(x, y) \in W$, where $\lambda \in [0, 1)$, $\alpha, \beta, L \in [0, +\infty)$ such that $\alpha < \beta$
Assume that:

(a) *W is Banach $T-$ invariant.*
(b) *if $\{u_n\}$ is a sequence in X such that $(u_{n-1}, u_n) \in W$ for all $n \in \mathbb{N}$ and $u_n \to u \in X$ as $n \to \infty$, then $(u_{n-1}, u) \in W$,$\forall n \in \mathbb{N}$*
(c) *$F(T)$ is well ordered with respect to W.*

If there exists $u_0 \in X$ such that $(u_0, Tu_0) \in W$ and (2) holds
Then,

(i) *T has a fixed point $u \in X$*
(ii) *for any $u_0 \in X$, the sequence $\{u_n\}$ converges to a fixed point of T*
(iii) *if $u, v \in X$ are two different fixed points of T, then*
$d(u, v) \geq max\{\frac{\beta(b-L-\lambda)-\alpha}{6}, 0\}.$

Proof. Let $\{u_n\}$ be a sequence in X with an initial approximation $u_0 \in X$ such that $Tu_n = u_{n+1}$.
Assume that $(u_0, Tu_0) \in W$, then (2) holds.

If $u_n = u_{n-1}$ for some $n \in \mathbb{N}$, then u_{n-1} is a fixed point of T .

Now we assume that $u_{n-1} \neq u_n \; \forall n \in \mathbb{N}$.

Since W is Banach $T-$ invariant, we get that $(u_1, u_2) = (Tu_0, T^2u_0) \in W$ for $(u_0, u_1) = (u_0, Tu_0) \in W$.

Using (1) with $x = u_{n-1}$ and $y = u_n \; \forall n \in \mathbb{N}$, we have

$$bd(u_n, u_{n+1}) = bd(Tu_{n-1}, Tu_n)$$
$$\leq \left(\frac{d(u_{n-1}, u_{n+1}) + d(u_n, u_n) + d(u_{n-1}, u_{n+2}) + d(u_n, u_{n+1}) + d(u_n, u_{n+2}) + d(u_{n+1}, u_{n+1}) + \alpha}{d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + d(u_{n-1}, u_{n+1}) + d(u_n, u_{n+2}) + d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \beta} + \lambda\right) d(u_{n-1}, u_n) + Ld(u_n, u_n)$$
$$\leq \left(\frac{d(u_{n-1}, u_{n+1}) + b(d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2})) + d(u_n, u_{n+1}) + d(u_n, u_{n+2}) + \alpha}{d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + d(u_{n-1}, u_{n+1}) + d(u_n, u_{n+2}) + d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \beta} + \lambda\right) d(u_{n-1}, u_n)$$
$$\leq \left(\frac{d(u_{n-1}, u_{n+1}) + bd(u_{n-1}, u_n) + bd(u_n, u_{n+1}) + bd(u_{n+1}, u_{n+2}) + d(u_n, u_{n+1}) + \alpha}{d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + d(u_{n-1}, u_{n+1}) + d(u_n, u_{n+2}) + d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \beta} + \lambda\right) d(u_{n-1}, u_n)$$
$$\leq b\left(\frac{d(u_{n-1}, u_{n+1}) + d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2})\alpha}{d(u_{n-1}, u_n) + d(u_n, u_{n+1}) + d(u_{n-1}, u_{n+1}) + d(u_n, u_{n+2}) + d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \beta} + \lambda\right) d(u_{n-1}, u_n)$$

Then,

$$d(u_n, u_{n+1}) \leq \left(\frac{(1+b)d(u_{n-1}, u_n) + (2b+2)d(u_n, u_{n+1}) + (1+b)d(u_{n+1}, u_{n+2}) + \alpha}{(1+b)d(u_{n-1}, u_n) + (2b+2)d(u_n, u_{n+1}) + (1+b)d(u_{n+1}, u_{n+2}) + \beta} + \lambda\right) d(u_{n-1}, u_n)$$

Since T is nonexpansive mapping, we get

$$d(u_n, u_{n+1}) \leq \left(\frac{(1+b)d(u_0, u_1) + (2b+2)d(u_1, u_2) + (1+b)d(u_2, u_3) + \alpha}{(1+b)d(u_0, u_1) + (2b+2)d(u_1, u_2) + (1+b)d(u_2, u_3) + \beta} + \lambda\right) d(u_{n-1}, u_n)$$
$$= \left(\frac{(1+b)d(u_0, Tu_0) + (2b+2)d(Tu_0, T^2u_0) + (1+b)d(T^2u_0, T^3u_0) + \alpha}{(1+b)d(u_0, Tu_0) + (2b+2)d(Tu_0, T^2u_0) + (1+b)d(T^2u_0, T^3u_0) + \beta} + \lambda\right) d(u_{n-1}, u_n)$$
$$\leq \left(\frac{(4+4b)d(u_0, Tu_0) + \alpha}{(4+4b)d(u_0, Tu_0) + \beta} + \lambda\right) d(u_{n-1}, u_n)$$
$$= \delta d(u_{n-1}, u_n)$$

Where

$$\delta = \frac{(4+4b)d(u_0, Tu_0) + \alpha}{(4+4b)d(u_0, Tu_0) + \beta} + \lambda < 1$$

Similary at the proof of Theorem 3.1

we obtain that $\{u_n\}$ is Cauchy sequence. By the completeness of X, there exists $x \in X$ such that $lim_{n \to \infty} u_n = u$.

From hypothesis (b) we have $(u_n, u) \in W$; taking $x = u_n$ and $y = u$ in (4), we get

$$bd(u_{n+1}, Tu) = bd(Tu_n, Tu)$$
$$\leq \left(\frac{d(u_n, Tu) + d(u, Tu_n) + d(u_n, T^2u) + d(u, T^2u_n) + d(Tu_n, T^2u) + d(Tu, T^2u_n) + \alpha}{d(u_n, Tu_n) + d(u, Tu) + d(u_n, T^2u_n) + d(u, T^2u) + d(Tu_n, T^2u_n) + d(Tu, T^2u) + \beta} + \lambda\right) d(u_n, u) + Ld(u, Tu_n)$$
$$\leq \left(\frac{d(u_n, Tu) + d(u, u_{n+1}) + d(u_n, T^2u) + d(u, u_{n+2}) + d(u_{n+1}, T^2u) + d(Tu, u_{n+2}) + \alpha}{d(u_n, u_{n+1}) + d(u, Tu) + d(u_n, u_{n+2}) + d(u, T^2u) + d(u_{n+1}, u_{n+2}) + d(Tu, T^2u) + \beta} + \lambda\right) d(u_n, u) + Ld(u, u_{n+1}) \quad (5)$$

Taking limit as $n \to \infty$ in (5) , we obtain that $d(u, Tu) \leq 0$ then $Tu = u$,in other words u is fixed point of T.Thus (i) and (ii) hold.

Let v b a another fixed point of T with $u \neq v$. Using (4) with $x = u$ and $y = v$, we get

$$bd(u, v) = bd(Tu, Tv)$$

$\leq (\frac{d(u,Tv)+d(v,Tu)+d(u,T^2v)+d(v,T^2u)+d(Tu,T^2v)+d(Tv,T^2u)+\alpha}{d(u,Tu)+d(v,Tv)+d(u,T^2u)+d(v,T^2v)+d(Tu,T^2u)+d(Tv,T^2v)+\beta}+\lambda)d(u,v)+$
$Ld(v,Tu)$
$=(\frac{6d(u,v)+\alpha}{\beta}+\lambda)d(u,v)+Ld(v,u)$

and hence $d(u,v)\geq \frac{\beta(b-L-\lambda)-\alpha}{6}$, that is (iii) holds.

□

If we take $\alpha=0,\beta=1$ in Theorems 3.1 and 3.2 we obtain the following results.

Corollary 2.6. *Let (X,d) be a complete rectangular $b-$ metric space with coefficient $b\geq 1$ endowed with a binary relation W on X*

and let $T:X\to X$ be a nonexpansive mapping such that

$$bd(Tx,Ty)\leq$$
$$(\frac{d(x,Ty)+d(y,Tx)+d(x,T^2y)+d(y,T^2x)+d(Tx,T^2y)+d(Ty,T^2x)}{d(x,Tx)+d(y,Ty)+d(x,T^2x)+d(y,T^2y)+d(Tx,T^2x)+d(Ty,T^2y)+1}+\lambda)d(x,y)$$

for all $(x,y)\in W$, where $\lambda\in[0,1)$

Assume that:

(a) *W is Banach $T-$ invariant.*
(b) *if $\{u_n\}$ is a sequence in X such that $(u_{n-1},u_n)\in W$ for all $n\in\mathbb{N}$ and $u_n\to u\in X$ as $n\to\infty$, then $(u_{n-1},u)\in W$,$\forall n\in\mathbb{N}$*
(c) *$F(T)$ is well ordered with respect to W.*

If there exists $u_0\in X$ such that $(u_0,Tu_0)\in W$ and

$$\frac{(4+4b)d(u_0,Tu_0)}{(4+4b)d(u_0,Tu_0)+1}+\lambda<1$$

Then,

(i) *T has a fixed point $u\in X$*
(ii) *for any $u_0\in X$, the sequence $\{u_n\}$ convergs to a fixed point of T*
(iii) *if $u,v\in X$ are two different fixed points of T, then*
$d(u,v)\geq max\{\frac{(b-\lambda)}{6},0\}.$

Corollary 2.7. *Let (X,d) be a complete rectangular $b-$ metric space with coefficient $b\geq 1$ endowed with a binary relation W on X*

and let $T:X\to X$ be a nonexpansive mapping such that

$$bd(Tx,Ty)\leq$$
$$(\frac{d(x,Ty)+d(y,Tx)+d(x,T^2y)+d(y,T^2x)+d(Tx,T^2y)+d(Ty,T^2x)}{d(x,Tx)+d(y,Ty)+d(x,T^2x)+d(y,T^2y)+d(Tx,T^2x)+d(Ty,T^2y)+1}+$$
$$\lambda)d(x,y)+Ld(y,Tx)$$

for all $(x,y)\in W$, where $\lambda\in[0,1)$

Assume that:

(a) *W is Banach $T-$ invariant.*
(b) *if $\{u_n\}$ is a sequence in X such that $(u_{n-1},u_n)\in W$ for all $n\in\mathbb{N}$ and $u_n\to u\in X$ as $n\to\infty$, then $(u_{n-1},u)\in W$,$\forall n\in\mathbb{N}$*
(c) *$F(T)$ is well ordered with respect to W.*

If there exists $u_0 \in X$ such that $(u_0, Tu_0) \in W$ and (2) holds

Then,

(i) *T has a fixed point $u \in X$*

(ii) *for any $u_0 \in X$, the sequence $\{u_n\}$ converges to a fixed point of T*

(iii) *if $u, v \in X$ are two different fixed points of T, then*

$$d(u,v) \geq max\{\frac{(b-L-\lambda)}{6}, 0\}.$$

If we take $W = X \times X$ in Theorems 3.1 and 3.2 we have the following results.

Theorem 2.8. *Let (X, d) be a complete rectangular $b-$ metric space with coefficient $b \geq 1$ and let $T : X \to X$ be a nonexpansive mapping such that*

$$bd(Tx,Ty) \leq$$
$$(\frac{d(x,Ty)+d(y,Tx)+d(x,T^2y)+d(y,T^2x)+d(Tx,T^2y)+d(Ty,T^2x)+\alpha}{d(x,Tx)+d(y,Ty)+d(x,T^2x)+d(y,T^2y)+d(Tx,T^2x)+d(Ty,T^2y)+\beta}+\lambda)d(x,y)$$

for all $(x, y) \in X$, where $\lambda \in [0, 1), \alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$.

If there exists $u_0 \in X$ such that

$$\frac{(4+4b)d(u_0,Tu_0)+\alpha}{(4+4b)d(u_0,Tu_0)+\beta}+\lambda < 1$$

Then,

(i) *T has a fixed point $u \in X$*

(ii) *for any $u_0 \in X$, the sequence $\{u_n\}$ converges to a fixed point of T*

(iii) *if $u, v \in X$ are two different fixed points of T, then*

$$d(u,v) \geq max\{\frac{\beta(b-\lambda)+\alpha}{6}, 0\}.$$

Theorem 2.9. *Let (X, d) be a complete rectangular $b-$ metric space with coefficient $b \geq 1$ and let $T : X \to X$ be a nonexpansive mapping such that*

$$bd(Tx,Ty) \leq$$
$$(\frac{d(x,Ty)+d(y,Tx)+d(x,T^2y)+d(y,T^2x)+d(Tx,T^2y)+d(Ty,T^2x)+\alpha}{d(x,Tx)+d(y,Ty)+d(x,T^2x)+d(y,T^2y)+d(Tx,T^2x)+d(Ty,T^2y)+\beta}+$$
$$\lambda)d(x,y)+Ld(y,Tx)$$

for all $(x, y) \in X$, where $\lambda \in [0, 1), \alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$.

If there exists $u_0 \in X$ such that

$$\frac{(4+4b)d(u_0,Tu_0)+\alpha}{(4+4b)d(u_0,Tu_0)+\beta}+\lambda < 1$$

Then,

(i) *T has a fixed point $u \in X$*

(ii) *for any $u_0 \in X$, the sequence $\{u_n\}$ converges to a fixed point of T*

(iii) *if $u, v \in X$ are two different fixed points of T, then*

$$d(u,v) \geq max\{\frac{\beta(b-L-\lambda)-\alpha}{6}, 0\}.$$

References

[1] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta mathematicae, 3(1), 133-181.

[2] Branciari, A. (2000). A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. Publ. Math. Debrecen, 57, 31–37.

[3] George, R., Radenovic, S., Reshma, K. P., & Shukla, S. (2015). Rectangular b-metric space and contraction principles. J. Nonlinear Sci. Appl, 8(6), 1005-1013.

[4] Kari, A., Rossafi, M., Marhrani, E. M., & Aamri, M. (2020). Fixed-point theorem for nonlinear F-contraction via w-distance. Advances in Mathematical Physics, 2020, 1-10.

[5] Kari, A., Rossafi, M., Saffaj, H., Marhrani, E. M., & Aamri, M. (2020). Fixed point theorems for $\theta - \phi -$contraction in generalized asymmetric metric spaces, International Journal of Mathematics and Mathematical Sciences, 2020, 1-19.

[6] Kari, A., Rossafi, M., Marhrani, E. M., & Aamri, M. (2020). New fixed point theorems for $\theta - \phi -$contraction on rectangular b-metric spaces, Abstract and Applied Analysis, (2020), , Article ID 8833214.

[7] Kari, A., Rossafi, M., Marhrani, E. M., & Aamri, M. (2020). $\theta - \phi -$contraction on $(\alpha, \eta)-$complete rectangular $b-$metric spaces, Int. J. Math. Mathematical Sciences, (2020), Article ID 5689458.

[8] Kari, A., Rossafi, M., Saffaj, H., Marhrani, E. M., & Aamri, M. (2022). Fixed Point Theorems for F-Contraction in Generalized Asymmetric Metric Spaces. Thai Journal of Mathematics, 20(3), 1149-1166.

[9] Rossafi, M., & Kari, A. (2022). Fixed points for weakly contractive mappings in rectangular b-metric spaces. International Journal of Nonlinear Analysis and Applications, (14), 763–783.

[10] Rossafi, M., Kari, A., Park, C., & Lee, J. R. (2023). New fixed point theorems for $\theta - \phi -$contraction on $b-$metric spaces. Journal of Mathematics and Computer Science, 29(1), 12-27.

[11] Rossafi, M., Kari, A., & Lee, J. R. (2022). Best Proximity Point Theorems For Cyclic $\theta - \phi -$contraction on metric spaces. J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. (29), 335–352

[12] Rossafi, M., Kari, A., & Lee, J. R. (2022). Fixed Point Theorems For $(\phi; F)-$Contraction in Generalized Asymmetric Metric Spaces. J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. 4 (29), 369–399