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Certain Orthogonality Conditions for Finite Elementary Operators in Normed Spaces

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ABSTRACT. Let H be an infinite dimensional complex Hilbert space and $B(H)$ be the space of all bounded linear operators on H. In this paper we give Certain Orthogonality Conditions for Finite Elementary Operators in Normed Spaces.

1. Introduction

Let H be an infinite dimensional complex Hilbert space and $B(H)$ be the space of all bounded linear operators on H. In this paper, We consider finiteness of elementary operators, orthogonality conditions for finite elementary operators and Birkhoff-James orthogonality for finite elementary operators. The following definitions are fundamental in the sequel:

Definition 1.1. Let X be a linear space over $\mathbb{F}$ .Then a norm on X is a non-negative real-valued function $\|.\| : X \to \mathbb{R}$ such that $\forall\ w, z \in X$ and $\eta \in \mathbb{F}$ the following properties are satisfied:

(i). $\|w\| \geq 0$ and $\|w\| = 0$, if and only if $w = 0$.
(ii). $\|\eta w\| = |\eta|\|w\|$.
(iii). $\|w + z\| \leq \|w\| + \|z\|$

The ordered pair $(X, \|.\|)$ is called a normed space.

Definition 1.2. A Banach space is a complete normed linear space.

Definition 1.3. Suppose Z is a real or complex valued vector space with an inner product $\langle ., . \rangle$. Then X is an inner product space.

Definition 1.4. A Hilbert space H is a complete inner product space.

Definition 1.5. Two vectors $w,\ z \in H$ are called orthogonal, denoted by $w \perp z$ if $\langle w, z \rangle = 0$.

Definition 1.6. Consider a normed space $\mathcal{D}$ and let $T{:}\mathcal{D} \to \mathcal{D}$. T is said to be an elementary operator if it can be represented in the following form $T(X) = \Sigma_{i=1}^{n} S_i X P_i$ for all $X \in \mathcal{D}$ where S_i and P_i are fixed in $\mathcal{D}$.

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Example 1.7. Let $S = L(Z)$ for $S, P \in L(Z)$ we define particular elementary operator.

(i). The left multiplication operator $L_S : L(Z) \to L(Z)$ by $L_S(X) = SX, \forall\, X \in L(Z)$.

(ii). The right multiplication operator$R_P : L(Z) \to L(Z)$ by $R_P(X) = XP, \forall\, X \in L(Z)$.

(iii). The generalized derivation by $\delta_{S,P} = L_S - R_P$.

(iv). The basic elementary operator by $M_{S,P}(X) = SXP, \forall\, X \in L(Z)$.

(iv). The Jordan elementary operator by $\mu_{S,P}(X) = SXP + PXS, \forall\, X \in L(Z)$.

Definition 1.8. The range of an operator $P : L(H) \to L(H)$ is defined as $Ran(T) = \{y \in L(H) : y = T(x)\ \forall\, x \in L(H)\}$.

Definition 1.9. The kernel of an operator $T : L(H) \to L(H)$ is defined as $Ker(T) = \{x \in L(H) : T(x) = 0\ \forall\, x \in L(H)\}$.

Definition 1.10. A bounded linear operator S on a Hilbert space H is called finite if $\|I - SX - XS\| \geq 1$ for each $X \in L(H)$.

2. Preliminary Results

In this section, We consider finiteness of elementary operators on normed spaces.

Proposition 2.1. *Let Ω be a normed space, then for $S \in \Omega$, $\sigma_p(S) \neq \emptyset$ if S is normaloid.*

Proof. Let $S \in \Omega$ be normaloid, then $\|S\| = r(S)$. This implies that there exist $\lambda \in \sigma_p(S)$ such that $|\lambda| = \|S\|$. It is known that $\sigma_p(S) \subseteq \sigma_{ap}(S) \subseteq \sigma(S)$. Therefore, $\sigma_p(S) = \sigma_{ap}(S)$. But λ is in the boundary of $\sigma_p(S)$ and since this is a subset of the approximate point spectrum of S, we have that $\lambda \in \sigma_p(S) = \sigma_{ap}(S)$. But for a sequence $\{x_n\}_{n\in N}$ of unit vectors we have, $\|(S - \lambda I)x_n\| \to 0$. So $0 \in \sigma_p(S)$ and hence $\sigma_p(S) \neq \emptyset$. □

Proposition 2.2. *Every normaloid operator is finite.*

Proof. From Proposition 2.1, we have that $\sigma_p(S) \neq \emptyset$ if S is normaloid. To show that every normaloid operator is finite, we let S to be a normaloid operator, i.e $\|S\| = r(S)$. Hence, there exist $\lambda \in \sigma_p(S)$ such that $|\lambda| = \|S\|$. By definition, an operator S in a normed space Ω is finite if $\|SX - XS - I\| \geq 1$, for all $X \in \Omega$. But $\|(T - \lambda I)x_n\| \to 0$ with $\|x_n\| = 1$. Since x_n is a normalized sequence we have,

$$\begin{aligned} \|SX - XS - I\| &= \|(S - \lambda I)X - X(S - \lambda I) - I\| \\ &\geq |\langle (S - \lambda I)X_{x_n,x_n}\rangle - \langle X(S - \lambda I)_{x_n,x_n}\rangle - I| \\ &\geq |\langle (S - \lambda I)X - X(S - \lambda I)\rangle_{x_n,x_n} - I| \\ &\geq |\langle (SX - XS)_{x_n,x_n}\rangle - I| \end{aligned}$$

Letting $n \to \infty$ we obtain $\|SX - XS - I\| \geq 1$. □

Lemma 2.3. *Let $S \in \Omega$ be normaloid and $S_o \in \Omega$ be norm-attainable such that $SS_o = S_oS$. Then for every $\eta \in \sigma_p(S_o)$, $\|S_o - (SX - XS)\| \geq |\eta|\ \forall X \in \Omega$.*

Proof. From [71], if $S_o \in \Omega$ is norm-attainable, then it is normal. So, we let $\eta \in \sigma_p(S_o)$ and M_η be the eigenspace associated with η. Since $SS_o = S_oS$, we have $SS_o^* = S_o^*S$ by Fulglede Putnam's theorem [41]. Hence M_η reduces both S and S_o. According to the decomposition of $H = M_\eta \oplus M_\eta^\perp$, we write S, S_o and X as follows:

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}, S_o = \begin{pmatrix} \eta & 0 \\ 0 & S_2 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

we have,

$$\begin{aligned} \|S_o - (SX - XS)\| &= \left\| \begin{pmatrix} \eta - S_1X_1 - X_1S_1 & * \\ * & * \end{pmatrix} \right\| \\ &\geq \|\eta - S_1X_1 - X_1S_1\| \\ &\geq |\eta| \left\| S_1(\frac{X_1}{\eta}) - (\frac{X_1}{\eta})S_1 \right\| \\ &\geq |\eta|. \end{aligned}$$

□

Lemma 2.4. *Every paranormal operator S in a C^* algebra Ω is finite.*

Proof. Let S be a paranormal operator, then S is normal i.e $S^*S = SS^*$. By Berberian theorem, it is known that, there exist a $*$-isometric isomorphism $\psi : \Omega \to \Omega$ that preserves order; So,

$$\begin{aligned} \|S\|^2 = \|SS^*\| = 1 &\leq \|(SX - XS) - I\| \\ &\leq \|\psi(SX - XS) - I\| \\ &\leq \|\psi(S)\psi(X) - \psi(X)\psi(S) - I\| \end{aligned}$$

If $S \in \Omega$ is an element of $F(H)$ such that $\sigma_p(S) \neq \emptyset$ then it results from Proposition 4.2 that $\psi(S) \in \Omega$ is finite i.e

$$\|SX - XS - I\| = \|\psi(S)\psi(X) - \psi(X)\psi(S) - I\| \geq 1.$$

□

Theorem 2.5. *Let $S \in \Omega$ be norm-attainable. Then $J = S + P$ is finite where P is compact in a C^*-algebra Ω.*

Proof. Let S be norm-attainable, since Ω is a C^*-algebra, it follows that $J = S + P$ is finite and we have,

$$\begin{aligned} \|J\|^2 = \|JJ^*\| = 1 &\leq \|(JX - XJ) - I\| \\ &\leq \|(SX - XS) - I\| \\ &\leq \|(SX + PP^{-1} - XS + P^{-1}P) - I\| \\ &\leq \|(S + P)(X + P^{-1}) - (X + P^{-1})(S + P) - I\|. \end{aligned}$$

For $Y = X + P^{-1}$ we have, $\|(S + P)Y - Y(S + P) - I\| \geq 1$. This proves that $J = S + P$ is a finite operator. □

Corollax 2.6. *Let $S \in \Omega$ be log-hyponormal and S^*be p-hyponormal then $\|J-(SX-XS_o)\| \geq \|J\|$ for all $X \in \Omega$ and for all $J \in ker\delta_{S,S_o}$.*

Proof. If $J \in$Kerδ_{S,S_o}, then also $J \in$Ker$\delta_{S^*S_o^*}$ by Putnam-Fuglede's theorem [41]. Therefore, $SJJ^* = JS_o^* = JJ^*S$. Since S is log-hyponormal, JJ^* is normal and $S(JJ^*) = (JJ^*)S$. Since $X \in \Omega$, we deduce that

$$\begin{aligned}\|J\|^2 = \|JJ^*\| &\leq \|JJ^* - SXJ^* - XJ^*S\| \\ &\leq \|JJ^* - SXJ^* - XS_oJ^*\| \\ &\leq \|J^*\|\|J-(SX-XS_o)\|\end{aligned}$$

This implies that $\|J\|^2 = \|J\|\|J^*\| \leq \|J^*\|\|J-(SX-XS_o)\|$.
Dividing both sides by $\|J^*\|$ we obtain,
$\|J\| \leq \|J-(SX-XS_o)\|$. □

3. Main Results

Remark 3.1. At this point, we characterize finiteness of elementary operators in a general set up. Let $\mathfrak{C}_n(S,S_o)$ be the set of all $(S,S_o) \in \Omega \times \Omega$ such that S and S_o have an n-dimensional reducing subspace $J_n(S,S_o)$ satisfying $S \mid J_n(S,S_o) = S_o \mid J_n(S,S_o)$.

Now we characterize finiteness in the cartesian product of $\Omega \times \Omega$ in the next proposition.

Proposition 3.2. *Let $(S,S_o) \in \mathfrak{C}_n(S,S_o)$. Then $\|(SX-XS_o)-I\| \geq 1$.*

Proof. Let $S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$ be the matrix representation of S and S_o respectively relative to the decomposition $H = H_1 \oplus H_1^\perp$ where H_1 is an n-dimensional reducing subspace of S and S_o i.e $H_1 = J_n(S,S_o)$. For any operator X on H has a representation $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$, where $K = I$. If we take $S_1X_1 - X_1S_2 = 0$, an easy calculation shows that,

$$\begin{aligned}\|(SX-XS_o)-I\| &= \left\| \left[\begin{pmatrix} S_1X_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\ &\geq \left\| \begin{pmatrix} S_1X_1 - X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\ &\geq \left\| \begin{pmatrix} (S_1X_1 - X_1S_2) - K_1 & K_2 \\ K_2 & K_2 \end{pmatrix} \right\| \\ &\geq \left\| \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\ &\geq \|K\|\end{aligned}$$

This implies that $\|(SX-XS_o)-I\| \geq \|K\|$ but since $K = I$ we have, $\|(SX-XS_0)-I\| \geq \|I\|$. Hence $\|(SX-XS_o)-I\| \geq 1$. □

Proposition 3.3. *Let $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. Then $\|SXS_o - I\| \geq 1$.*

Proof. Let S, S_o, X and K have the following representation: $S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$, $K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$, where $K = I$. Let $S_1X_1S_2 = 0$, it follows that,

$$\begin{aligned}
\|SXS_o - I\| &= \left\| \left[\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} S_1X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} (S_1X_1S_2) - K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \|K\|
\end{aligned}$$

This implies that $\|SXS_o - I\| \geq \|K\|$ but since $K = I$ we have, $\|SXS_0 - I\| \geq \|I\|$. Hence $\|SXS_o - I\| \geq 1$. □

Theorem 3.4. *Let $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. Then $\|(SXS_o + S_oXS) - I\| \geq 1$.*

Proof. Let $S, S_o, X,$ and K have the following representation[decomposition] $S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$, $K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$, where $K = I$ and Let $S_1X_1S_2 + S_2X_1S_1 = 0$. We show that $\|(SXS_o + S_oXS) - I\| \geq 1$. So,

$$\begin{aligned}
\|(SXS_o + S_oXS) - I\| &= \left\| \left[\begin{pmatrix} S_1X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} S_2X_1S_1 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} S_1X_1S_2 + S_2X_1S_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} (S_1X_1S_2 + S_2X_1S_1) - K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \|K\|
\end{aligned}$$

This implies that $\|(SXS_o + S_oXS) - I\| \geq \|K\|$ but since $K = I$ we have, $\|(SXS_o + S_oXS) - I\| \geq \|I\|$. Thus $\|(SXS_o + S_oXS) - I\| \geq 1$. □

Theorem 3.5. *Let $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. Then $\|(SXS_o + CXC_o) - I\| \geq 1$.*

Proof. Let S, S_o, C, C_o, X, and K have the following representation[decomposition] $S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$,$C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}$,$C_o = \begin{pmatrix} C_2 & 0 \\ 0 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$, where $K = I$. Let $S_1X_1S_2 + C_1X_1C_2 = 0$. Then we have,

$$\begin{aligned}
\|(SXS_o + CXC_o) - I\| &= \left\| \left[\begin{pmatrix} S_1X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} C_1X_1C_2 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} S_1X_1S_2 + C_1X_1C_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} (S_1X_1S_2 + C_1X_1C_2) - K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \|K\|
\end{aligned}$$

This implies that $\|(SXS_o + CXC_o) - I\| \geq \|K\|$ but $K = I$ hence we have, $\|(SXS_o + CXC_o) - I\| \geq \|I\|$. Thus $\|(SXS_o + CXC_o) - I\| \geq 1$. □

Theorem 3.6. *Let $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. Then $\| \sum_{i=1}^n S_iXC_i - I\| \geq 1$*

Proof. Let S, X, S_o and K have the following representation. $S_i = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$, $C_i = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}$, $K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$, where $K = I$. If we take $\sum_{i=1}^n S_1X_1C_1 = 0$, it follows that,

$$\begin{aligned}
\| \sum_{i=1}^n S_iXC_i - I\| &= \left\| \sum_{i=1}^n \left[\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \sum_{i=1}^n \begin{pmatrix} S_1X_1C_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right\| \\
&\geq \|K\|
\end{aligned}$$

This implies that $\| \sum_{i=1}^n S_iXC_i - I\| \geq \|K\|$ but since $K = I$ we have, $\| \sum_{i=1}^n S_iXC_i - I\| \geq \|I\|$. Hence $\| \sum_{i=1}^n S_iXC_i - I\| \geq 1$. □

Remark 3.7. It is known [78] that there exists a compact operator C such that $R(\delta_c) = K(H)$. As a consequence we deduce that the $\text{dist}(I, K(H)) = 1$, where $\text{dist}(I, K(H))$ is the distance from I to $K(H)$. Therefore if S, S_o are compact operators, then $\text{dist}(I, R(\delta_{S,S_o})) = 1$. The previous corollary shows that $\mathfrak{C}_n(S, S_o) \subset F(H)$. Hence it is interesting to pose the following open problem.

Open problem. Does $F(H) \subset \mathfrak{C}_n(S, S_o)$?

3.1. **Orthogonality conditions for finite elementary operators.** Let Ω denote a Complex Banach algebra with identity e and let $\sigma_r(\Omega)$, $\sigma_l(\Omega)$ denote, respectively the right spectrum and the left spectrum of Ω. Recall that

$$S^n X - X S_o^n = \sum_{i=0}^{n-i-1} (SX - XS_o) S_o^i \text{ for all } X \in \Omega.$$

If $SJ = JS_o$we have,

$$nJS_o^{n-1} = S^n X - XS_o^n - \sum_{i=0}^{n-i-1} S^{n-i-1}(SX - XS_o - J)S_o^i \text{ for all } X \in \Omega.$$

Proposition 3.8. *Let $S \in \Omega$,(x_n) be an increasing sequence of positive integers and S^{x_n} converge to $Z \in \Omega$, with $0 not \in \sigma_r(Z) \cap \sigma_l(Z)$. If there exist a constant λ such that $\|S^n\| \leq \lambda$ for all integers n and if S_o is the left or right inverse of Z then*

$$\lambda^2 \|S_o\| \|SX - XS - J\| \geq \|J\| \textit{ for all } X \in \Omega \textit{ and for all } J \in Ker\delta_S.$$

Proof. Let $X \in \Omega$, since

$$nJS_o^{n-1} = S^n X - XS_o^n - \sum_{i=0}^{n-i-1} S^{n-i-1}(SX - XS_o - J)S_o^i \text{ for } SJ = JS_o.$$

we can write

$$\begin{aligned}
(x_n+1)JS^{x_n+1-1} &= S^{x_n+1}X - XS^{x_n+1} - \sum_{i=0}^{x_n+1-i-1} S^{x_n+1-i-1}(SX - XS - J)S^i \\
&= S^{x_n+1}X - XS^{x_n+1} - \sum_{i=0}^{x_n-i} S^{x_n-i}(SX - XS - J)S^i
\end{aligned}$$

Now the assumption that $\|S^n\| \leq \lambda$ implies that $\|S^{x_n+1}\| \leq \lambda$ for all $n \in \Omega$.Dividing both sides by $x_n + 1$ and taking norms we obtain

$$\begin{aligned}
\|JS^{x_n}\| &\leq \frac{1}{x_n+1} \||S^{x_n+1}| + |S^{x_n+1}|\|\|X\| + \lambda^2 \|SX - XS - J\| \\
&\leq \frac{2\lambda}{x_n+1} \|X\| + \lambda^2 \|SX - XS - J\|
\end{aligned}$$

Letting tend to infinity we obtain

$$\|JS^{x_n}\| \leq \lambda^2 \|SX - XS - J\|$$

But S^{x_n} converges to Z, so we have,

$$\|JZ\| \leq \lambda^2 \|SX - XS - J\|$$

Now, since S_o is in the right or the left of Z we have

$$\|J\| \leq \|S_o\| \lambda^2 \|SX - XS - J\|.$$

□

Corollax 3.9. *Let $S \in L(H)$ and (x_n) be an increasing sequence of positive integers. Assume that there is a constant λ such that $\|S^n\| \leq \lambda$ for all integers n*

1).If $S_{x_n} \to P$, with $0 \in \sigma_r(P) \cap \sigma_l(P)$, then $\lambda^2\|SX - XS - J\| \geq \|J\|$ for all $X \in L(H)$ and for all $J \in Ker\delta_S$. 2).If $S_{x_n} \to P + K$, with K compact and $0 \in \sigma_r(P) \cap \sigma_l(P)$, then $\lambda^2\|SX - XS - J - K\| \geq \|J\|$ for all $X \in L(H)$ and for all $J \in Ker\delta_S$.

Theorem 3.10. *Let $S \in L(H)$ such that $S^n = I$ for some integer n. Then*

$$\lambda^2\|SX - XS - J\| \geq \|J\| \text{ for all } X \in \Omega \text{ and for all } J \in Ker\delta_S.$$

Proof. Since $S^n = \{I, S, S^2, ...S^{m-1}\}$ for all integers n, $\|S^n\| \leq \lambda$, $n \in \mathbb{N}$ and $S^{x_n} = I$, where $x_n = nm$, $n \in \mathbb{N}$. It is known that

$$nJS_o^{n-1} = S^nX - XS_o^n - \sum_{i=0}^{n-i-1} S^{n-i-1}(SX - XS_o - J)S_o^i \text{ for all} X \in \Omega.$$

From proposition 1 we have that

$$(x_n+1)JS^{x_n+1-1} = S^{x_n+1}X - XS^{x_n+1} - \sum_{i=0}^{x_n+1-i-1} S^{x_n+1-i-1}(SX - XS - J)S^i //$$
$$(x_n+1)JS^{x_n} = S^{x_n+1}X - XS^{x_n+1} - \sum_{i=0}^{x_n-i} S^{x_n-i}(SX - XS - J)S^i$$

Dividing both sides by $x_n + 1$ and taking the norms we obtain

$$\begin{aligned} \|JS^{x_n}\| &\leq \frac{1}{x_n+1}\||S^{x_n+1}| + |S^{x_n+1}|\|\|X\| + \lambda^2\|SX - XS - J\| \\ &\leq \frac{2\lambda}{x_n+1}\|X\| + \lambda^2\|SX - XS - J\| \end{aligned}$$

Since $S^{x_n} = I$ we have

$$\|J\| \leq \tfrac{2\lambda}{x_n+1}\|X\| + \lambda^2\|SX - XS - J\|$$

Letting n tend to infinity,we get

$$\|J\| \leq \lambda^2\|SX - XS - J\|$$

Hence, $\|J\| \leq \lambda^2\|SX - XS - J\|$. □

Corollax 3.11. *Let $S_1, S_o \in L(H)$ such that $S_1^m = I$ and $S_o^m = I$ for some integer m. Then*

$$\|S_1X - XS_o - J\| \geq \|J\| \text{ for all } X \in \Omega \text{ and for all } J \in Ker\delta_{S_1,S_o}.$$

Proof. Consider the operators P, S and Y defined on $H \oplus H$ $P = \begin{pmatrix} S_1 & 0 \\ 0 & S_o \end{pmatrix}$, $S = \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ then P is normal on $H \oplus H$ and it is clear that $P^m = 1$, $PS = SP$ i.e $S \in Ker\delta_p$. Since $PY - YP = \begin{pmatrix} 0 & S_1X \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & S_oX \\ 0 & 0 \end{pmatrix}$

$$\begin{aligned} PY - YP - S &= \begin{pmatrix} 0 & S_1X - XS_o \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & S_1X - XS_o - J \\ 0 & 0 \end{pmatrix} \end{aligned}$$

□

Then it follows that

$$\|PY - YP - S\| \geq \|S\|$$

Consequently, we obtain

$$\|S_1X - XS_o - J\| = \|PY - YP - S\| \geq \|S\| = \|J\|.$$

Proposition 3.12. *Let $S, S_o \in F(H)$. If $S_o \in [F(H)]^{-1}$ and $\|S\|\|S_o^{-1}\| \leq 1$, then $\|\delta_{S,S_o} + J\| \geq \|J\|$ for all $X \in F(H)$ and $J \in Ker\delta_{S,S_o}$.*

Proof. Let $J \in F(H)$ such that $SJ = JS_o$. Therefore, $SJS_o^{-1} = J$. But $\|S\|\|S_o^{-1}\| = 1$. It follows from [] theorem 2.1 that

$$\|SYS_o - Y + J\| \geq J \,\forall Y \in F(H).$$

If we set $X = YS_o^{-1}$ then we obtain $\|SX - XS_o + J\| \geq \|J\| \;\forall Y \in F(H)$ But $SX - XS_o$=$\delta_{S,S_o}(X)$ Hence $\|\delta_{S,S_o}(X) + J\| \geq \|J\|$ for all $J \in Ker\delta_{S,S_o}$ and for all $Y \in F(H)$. □

Remark 3.13. If $(J, \|.\|_J)$ is a norm ideal then the norm $\|.\|_J$ is unitarily invariant in the sense that $\|SXP\|_J = \|T\|J$ for all $T \in J$ and for all unitary operators.

Corollax 3.14. *Let $(J, \|.\|_J)$ be a norm ideal and $S, P \in L(H)$.If S is an isometry and the operator P is a contraction, then*

$$\|\delta_{S,P}(X) + T\|_J \geq \|T\|_J \textit{ for all } X \in J \textit{ and for all } J \in Ker\delta_{S,P}.$$

Proposition 3.15. *Let $(J, \|.\|_J)$ be a norm ideal and $S \in F(H)$. Suppose that $F(S)$ is a cyclic subnormal operator , where f is a nonconstant analytic function on an open set containing $\sigma(S)$. Then*

$$\|\delta_S(X) + T\|_J \geq \|T\|_J \textit{ for all } X \in J \textit{ and for all } J \in \{S\} \cap J.$$

Proof. Let $T \in J$ such that $ST = TS$. This implies that $Tf(S) = f(S)T$ and $Sf(S) = f(S)S$. Since $F(S)$ is a cyclic subnormal operator, it follows from [.] that S and T are subnormal. But every subnormal operator is hyponormal [.]. Therefore, T is normal. Consequently, $ST = TS$ implies that $ST^* = T^*S$ by Putnam-Fuglede Theorem. Hence, $\overline{R(T)}$ and $Ker(T)^\perp$ reduces $S$and $S\mid_{\overline{R(T)}}$ and $S\mid_{Ker(T)^\perp}$ are normal operators. Let $T_ox = T_x$ for each $x \in Ker(T)$, it follows that $\delta_{S,P}(T_o) = \delta_{S^*,P^*}(T_o) = 0$. Let $S = S_1 \oplus S_2$ with respect to $H = \overline{R(T)} \oplus \overline{R(T)}^\perp$ and $P = P_1 \oplus P_2$ with respect to $H = \overline{Ker(T)^\perp} \oplus \overline{Ker(T)}$. Then we can write S, T and X as follows $S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} and X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$. Then,

$$\|(SX - XS) + T\|_J = \left\| \begin{pmatrix} S_1X_1 - X_1S_1 + T_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|_J$$

This implies that

$$\|(SX - XS) + T\|_J = \|S_1X_1 - X_1S_1 + T_1\|_J \geq \|T_1\|_J = \|T\|_J.$$

Hence, $\|\delta_S(X) + T\|_J = \|\delta_{S_1}(X) + T_1\|_J \geq \|T_1\|_J = \|T\|_J$. □

Proposition 3.16. *Let $S, P \in F(H)$ such that the pair (S, P) possesses the $PF(\delta, J)$ property. Then, $\|\delta_{S,P} + T\| \geq \|T\|$ for all $X \in J$ and $T \in Ker\delta_{S,P}$.*

Proof. Let $T \in J$, since the pair S, P satisfies $PF(\delta, J)$ property. Then $\overline{R(T)}$ reduces S and $Ker(T)^{\perp}$ reduces $P$and $S\ |_{\overline{R(T)}}$ and $P\ |_{Ker(T)^{\perp}}$ are normal operators. Let $T_o : Ker(T)^{\perp} \to \overline{R(T)}$ be the quasi affinity defined by setting $T_o x = T_x$ for each $x \in Ker(T)$, it results that $\delta_{S,P}(T_o) = \delta_{S^*,P^*}(T_o) = 0$. Let $S = S_1 \oplus S_2$ with respect to $H = \overline{R(T)} \oplus \overline{R(T)}^{\perp}$ and $P = P_1 \oplus P_2$ with respect to $H = \overline{Ker(T)^{\perp}} \oplus \overline{Ker(T)}$. Let S, P,T and X have the following representation. $S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} andX = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$. we have,

$$\|(SX - XP) + T\|_J = \left\| \begin{pmatrix} S_1X_1 - X_1S_1 + P_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|_J$$

This implies that

$$\|(SX - XS) + T\|_J = \|S_1X_1 - X_1P_1 + T_1\|_J \geq \|T_1\|_J = \|T\|_J.$$

Hence, $\|\delta_{S,P}(X) + T\|_J = \|\delta_{S_1,P_1}(X) + T_1\|_J \geq \|T_1\|_J = \|T\|_J$. □

Proposition 3.17. *Let $S, P \in F(H)$ be normal operators such that $SP = PS$ and$S^*S + P^*P > 0$. For an elementary operator $E(X) = SXP - PXS$, $|||E(X) + J||| \geq |||J|||$ for all $J \in KerE$.*

Proof. Assume that $P^{-1} \in L(H)$, then from $SP = PS$ and $SJP = PJS$ we get,$SP^{-1}J = JP^{-1}S$. Hence applying theorem AK [] to the operators SP^{-1}, $P^{-1}S$ and J we get,

$$\|(SX - XS) + J\| \geq \|SP_1PXP - PXP_1S + J\|_J \geq \|J\|_J.$$

Consider now the case when P is injective i.e $KerP = 0$. Let $\sigma_n = \{\lambda \in \mathbb{C} : \lambda \leq \frac{1}{n}\}$ and let $E_P(\sigma_n)$ be the corresponding spectral projector. If we put $P_n = I - E_P(\sigma_n)$. The subspace P_nH reduces both S and P (since they commute and are normal).Hence with respect to the decomposition$H = (I - P_n)H \oplus P_n(H)$ $S = \begin{pmatrix} 0 & 0 \\ 0 & S_1^{(n)} \end{pmatrix}, P = \begin{pmatrix} 0 & 0 \\ 0 & P_1^{(n)} \end{pmatrix} J = \begin{pmatrix} J_{11}^{(n)} & J_{12}^{(n)} \\ J_{21}^{(n)} & J_{22}^{(n)} \end{pmatrix} andX = \begin{pmatrix} X_{11}^{(n)} & X_{12}^{(n)} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{pmatrix}$. It is easy to see that $P_1^{(n)}$ acting on $P_n(H)$ is invertible. It follows that

$$\begin{aligned} |||SXP - PXS + J||| &\geq |||P_n(SXP - PXS + J)P_n||| \\ &= |S_1^{(n)}X_{22}^{(n)}P_1^{(n)} - P_1^{(n)}X_{22}^{(n)}S_1^{(n)} + J||| \\ &\geq |||J_{22}||| = |||P_nJP_n||| \end{aligned}$$

Therefore we have

$$|||SXP - PXS + J||| \geq |||P_nJP_n|||$$

applying Lemma 3[] we obtain

$$|||SXP - PXS + J||| \geq |||J|||$$

Now we assume that$KerS \cap KerP = \{0\}$. Let S, P, J and X have the following representation with respect to the space decomposition $H = KerP \oplus H_o(H_o \ominus KerP)$ $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, P = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix} J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} andX = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$. Operators S_1 and P_2 are injective and we have $(SXP - PXS) = \begin{pmatrix} 0 & S_1X_{12}P_2 \\ -P_2X_{21}S_1 & S_2X_{22}P_2 - P_2X_{22}S_2 \end{pmatrix}$ Since $SJP = PJS = 0$, then $S_2J_{22}P_2 = P_2J_{22}S_2$ and $S_1J_{12}P_2 = P_2J_{21}S_1 = 0$ since S_1 and P_2 are injective and their ranges are dense. So,

$$\begin{aligned} \|SXP - PXS + J\| &= \begin{pmatrix} 0 & S_1X_{12}P_2 \\ -P_2X_{21}S_1 & S_2X_{22}P_2 - P_2X_{22}S_2 \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \\ &= \begin{pmatrix} J_{11} & S_1X_{12}P_2 \\ -P_2X_{21}S_1 & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} \end{aligned}$$

Since P_2 is injective, we have already shown that $|||S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22}||| \geq |||J_{22}|||$ Applying LemmaGK[] and Lemma 2[] we have

$$\begin{aligned} |||S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22}||| &\geq \left| \left\| \begin{pmatrix} J_{11} & 0 \\ 0 & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} \right\| \right| \\ &\geq \left| \left\| \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix} \right\| \right| = |||J||| \end{aligned}$$

□

Theorem 3.18. *Let $S, P \in L(H)$ be normal operators such that $PS = SP$ and $E(X) = SXP - PXS$. If $J \in KerE$ then*

$$|||E(X) + J||| \geq 3^{-1}|||J|||$$
$$\|E(X) + J\|_p \geq 2^{|1-\frac{2}{P}|}\|J\|_P$$
where $\|.\|_P$ is the C_P norm and
$$\|E(X) + J\|_2^2 \geq \|J\|_2^2 + \|E(X)\|_2^2 \text{ where } \|J\|_2^2 \text{ is the Hilbert Schmidt norm.}$$

Proof. Let S, P, J and X have the following representation with respect to the space decomposition $H = H_1 \oplus H_2$ where $H_1 = KerS \cap KerP)$ and $H_2 = H \ominus H_1$ $S = \begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}, P = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix} J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} andX = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$. We have that $KerS_2 \cap KerP_2 = \{0\}$ in H_2. Applying Lemma 1 [] and proposition(above) we have

$$\begin{aligned} |||SXP - PXS + J||| &= \left| \left\| \begin{pmatrix} 0 & 0 \\ 0 & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \right\| \right| \\ &= \left| \left\| \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} \right\| \right| \end{aligned}$$

$$\begin{aligned} &\geq |||S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22}|| \,| \\ &= 2^{-1}\,|||S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22}||\,| \\ &= 2^{-1}\,|||J_{22}||\,| \end{aligned}$$

For us to prove [2] we start with the same inequalities before and then we apply Lemma$K$[] twice and proposition[above]. For $1 \leq P \leq 2$

$$\begin{aligned} |||SXP - PXS + J|||_P^P &= \left\| \begin{pmatrix} 0 & 0 \\ 0 & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \right\|_P^P \\ &= \left\| \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} + J_{22} \end{pmatrix} \right\|_P^P \\ &= 2^{p-2}(\|J_{11}\|_P^P + \|J_{12}\|_P^P + \|J_{21}\|_P^P + \\ \|S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} + J_{22}\|_P^P) \\ &= 2^{p-2}(\|J_{11}\|_P^P + \|J_{12}\|_P^P + \|J_{21}\|_P^P + \|J_{22}\|_P^P) \\ &= 2^{p-2}\|J\|_P^P \end{aligned}$$

and for $2 \leq P < \infty$ we have

$$\begin{aligned} |||SXP - PXS + J|||_P^P &= \left\| \begin{pmatrix} 0 & 0 \\ 0 & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \right\|_P^P \\ &= \left\| \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} + J_{22} \end{pmatrix} \right\|_P^P \\ &= 2^{2-P}(\|J_{11}\|_P^P + \|J_{12}\|_P^P + \|J_{21}\|_P^P + \\ \|S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} + J_{22}\|_P^P) \\ &= 2^{2-P}(\|J_{11}\|_P^P + \|J_{12}\|_P^P + \|J_{21}\|_P^P + \|J_{22}\|_P^P) \\ &= 2^{2-P}\|J\|_P^P \end{aligned}$$

Hence, $\|SXP - PXS + J\|_p \geq 2^{|1-\frac{2}{P}|}\|J\|_P^P$ and this proves equation 2 Now if $P = 2$ equation 2 becomes $\|E(X) + J\|_2 \geq \|J\|_2$ and according to Remark 1[]this implies 8 □

4. Conclusion

The main results of finiteness of elementary operators, orthogonality conditions for finite elementary operators and Birkhoff-James for finite elementary operators have been given in this paper.

5. Recommendations

The results obtained are specific to finiteness of elementary operators, orthogonality conditions for finite elementary operators and Birkhoff-James for finite elementary operators in complex normed spaces.

Open problem. Is $F(H) \subset \mathfrak{C}_n(S, S_o)$ in a general Banach space setting?

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