Functional Analysis
Volume 2, 2023, ID 4

FAC – SEAMS

Some Results of Fixed Point Theorems in Convex C^*-Algebra Valued Metric Spaces

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ABSTRACT. In this present article, we study some fixed point theorems for self mappings satisfying certain contraction principles on convex C^*-algebra valued complete metric space, we also improve some common fixed point theorems for a Banach operator pair on convex C^*-algebra valued complete metric space.

1. Introduction and preliminaries

Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, C^*-algebra valued metric spaces were introduced by Ma et al. [7] as a generalization of metric spaces.

In 1970 Takahashi [9] defined convexity and convex metric space to characterize a metric space, and studied some fixed point theorems for nonexpansive mappings in such spaces. It should be noted that every normed space and cone Banach space is a convex metric space and convex complete metric space, respectively. Chang, Kim and Jin [3], Ciric [5] and Shimizu and Takahashi [8], and many others studied fixed point theorems in convex metric spaces.

In this paper, we study the existence of a fixed point for self- mappings defined on a nonempty closed convex subset of a C^*-algebra valued convex complete metric space. Ours results extend some of Karapinar's results in [6] from a cone Banach space to a C^*-algebra valued convex complete metric space.

Definition 1.1. [1] Let (X, d) be a metric space and $I = [0, 1]$. A mapping $\mathcal{S} : X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$

$$d(u, \mathcal{S}(x, y, \lambda)) \leq (1 - \lambda)d(u, x) + \lambda d(u, y).$$

A metric space (X, d) with a convex structure $\mathcal{S}$ is called convex metric space and is denoted by $(X, \mathcal{S}, d)$. A subset $\mathcal{V}$ of X is called convex if $\mathcal{S}(x, y, \lambda) \in \mathcal{V}$ whenever $x, y \in \mathcal{V}$ and $\lambda \in I$.

Date submitted: 2023-04-09.
Key words and phrases. Fixed point, C^*-algebra valued metric spaces, convex C^*-algebra valued complete metric space, W-convex, Banach operator pair, common fixed point.

Definition 1.2. [1] Let $(X, d, \mathcal{S})$ be a convex metric space. A nonempty subset $\mathcal{V}$ of X is said to be convex if $\mathcal{S}(x, y, \lambda) \in \mathcal{V}$ whenever $(x, y, \lambda) \in \mathcal{V} \times \mathcal{V} \times I$.

Definition 1.3. [2] Let $(X, d, \mathcal{S})$ be a convex metric space and $\mathcal{V}$ be a convex subset of X. A self-mapping $T : \mathcal{V} \to \mathcal{V}$ has a property (I),
if $T(\mathcal{S}(x, y, \lambda)) = \mathcal{S}(Tx, Ty, \lambda)$ for each $(x, y, \lambda) \in \mathcal{V} \times \mathcal{V} \times I$.

Definition 1.4. [4] The ordered pair (T, R) of two self maps of a metric space (X, d) is called Banach operator pair if $T(Fix(R)) \subset Fix(R)$.

$Fix(T)$ and $Fix(T, R)$ denote the set of all fixed points of T and common fixed points of the pair (T, R), respectively.

Throughout this paper, we denote $\mathbb{A}$ by an unital (i.e, unity element I) C^*-algebra with linear involution $*$, such that for all $x, y \in \mathbb{A}$,

$$(xy)^* = y^* x^*, \text{ and } x^{**} = x.$$

We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$,
if $x \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x)$ is the spectrum of x.
Using positive element, we can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows :

$$x \preceq y \text{ if and only if } y - x \succeq \theta$$

where θ means the zero element in $\mathbb{A}$. We denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ by $\mathbb{A}_+$ and $|x| = (x^*x)^{\frac{1}{2}}$, $\mathbb{A}'$ will denote the set $\{a \in \mathbb{A}_+; ab = ba, \forall b \in \mathbb{A}\}$.

In [7], the authors introduced the concept of C^*- algebra valued metric spaces. The main idea is in using the set of all positive elements of a unital C^*- algebra instead of set of real numbers. Such spaces generalize the concept of metric spaces. In this paper, we give some fixed point theorems for self mappings defined on nonempty closed convex subset of a convex complete metric space with contractive condition on C^*- algebra valued convex metric spaces.

Definition 1.5. [7] Let X be nonempty set. Suppose the mapping : $d : X \times X \to \mathbb{A}$ satisfies the following conditions for each $x,\ y,\ z \in X$:

(1) $d(x, y) \succeq \theta$ and $d(x, y) = \theta \Leftrightarrow x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \preceq d(x, y) + d(y, z)$.

Then d is called a C^*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

Definition 1.6. [2] Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and $I = [0, 1]$. A mapping $\mathcal{S} : X \times X \times I \to X$ is said to be a convex structure on X, if $\forall x, y \in X$ and $\forall \lambda \in I$

$$d(z, \mathcal{S}(x, y, \lambda)) \preceq (1 - \lambda)d(z, x) + \lambda d(z, y), \forall z \in X \tag{1.1}$$

$(X, \mathbb{A}, d)$ with a convex structure $\mathcal{S}$ is called a convex C^*-algebra valued metric space and is denoted by $(X, \mathbb{A}, d, \mathcal{S})$.

A subset $\mathcal{V}$ of X is called convex if $\mathcal{S}(x, y, \lambda) \in \mathcal{V}$ whenever $x, y \in \mathcal{V}$ *and* $\lambda \in I$.

Example 1.7. Let $(X, \mathbb{A}, d, \mathcal{S})$ be a convex C^*-algebra valued metric space with $X = \mathbb{R}$, $\mathbb{A} = \mathbb{R}^2$ and $d : X \times X \to \mathbb{R}^2$ defined by $d(x, y) = (|x - y|, 0)$

and the mapping $\mathcal{S} : X \times X \times [0, 1] \to X$ defined by $\mathcal{S}(x, y, \lambda) = \lambda x + (1 - \lambda)y$.

We have for all $x, y, z \in X$

$$\begin{aligned} |z - \mathcal{S}(x, y, \lambda)| &= |z - (\lambda x + (1 - \lambda)y)| \\ &= |\lambda z - \lambda x + (1 - \lambda)(z - y)| \\ &\leq |\lambda(z - x)| + |(1 - \lambda)(z - y)| \\ &= \lambda|z - x| + (1 - \lambda)|z - y|. \end{aligned}$$

Therefore, $d(z, \mathcal{S}(x, y, \lambda)) \preceq (1 - \lambda)d(z, x) + \lambda d(z, y)$.

Then $(X, \mathbb{A}, d, \mathcal{S})$ is a convex C^*-algebra valued metric space.

2. Main result

Lemma 2.1. *Let $(X, \mathbb{A}, d, \mathcal{S})$ be a convex C^*-algebra valued metric space and $\lambda \in I$, we have*

$d(x, \mathcal{S}(x, y, \lambda)) = \lambda d(x, y)$,

$d(y, \mathcal{S}(x, y, \lambda)) = (1 - \lambda)d(x, y)$.

Proof. Using 1.1 we get

$$d(x, \mathcal{S}(x, y, \lambda)) \preceq \lambda d(x, y)$$

and

$$d(y, \mathcal{S}(x, y, \lambda)) \preceq (1 - \lambda)d(x, y)$$

we also have

$$\begin{aligned} &d(x, y) \preceq d(x, \mathcal{S}(x, y, \lambda)) + d(\mathcal{S}(x, y, \lambda), y) \\ &\Rightarrow d(x, y) \preceq (1 - \lambda)d(x, y) + \lambda d(x, y) \\ &\Rightarrow d(x, y) = d(x, y). \end{aligned}$$

Then

$$d(x, \mathcal{S}(x, y, \lambda)) + d(y, \mathcal{S}(x, y, \lambda)) = d(x, y).$$

Suppose that

$$d(x, \mathcal{S}(x, y, \lambda)) \prec \lambda d(x, y)$$

we obtain

$$d(x, \mathcal{S}(x, y, \lambda)) + d(\mathcal{S}(x, y, \lambda), y) \prec d(x, y)$$

which a contradiction. Therefore $d(x, \mathcal{S}(x, y, \lambda)) = \lambda d(x, y)$ and consequently $d(y, \mathcal{S}(x, y, \lambda)) = (1 - \lambda)d(x, y)$.

□

Corollary 2.2. *Let $(X, \mathbb{A}, d, \mathcal{S})$ be a convex C^*-algebra valued metric space, then we have*

(1) $d(x, \mathcal{S}(x, y, \lambda)) + d(y, \mathcal{S}(x, y, \lambda)) = d(x, y), \forall (x, y, \lambda) \in X \times X \times I$

(2) $d(x, \mathcal{S}(x, y, \frac{1}{2})) = d(y, \mathcal{S}(x, y, \frac{1}{2})) = \frac{1}{2}d(x, y), \forall x, y \in X.$

Proof. To prove (1), for any $(x,y,\lambda) \in X \times X \times I$, we have

$$\begin{aligned} d(x,y) &\preceq d(x,\mathcal{S}(x,y,\lambda)) + d(y,\mathcal{S}(x,y,\lambda)) \\ &\preceq (1-\lambda)d(x,y) + \lambda d(x,y) \\ &= d(x,y). \end{aligned}$$

(2) Let $x,y \in X$, by (1), we have

$$d(x,\mathcal{S}(x,y,\frac{1}{2}) \preceq \frac{1}{2}d(x,y) = \frac{1}{2}d(x,\mathcal{S}(x,y,\frac{1}{2})) + \frac{1}{2}d(y,\mathcal{S}(x,y,\frac{1}{2})).$$

This implies that

$$\frac{1}{2}d(x,\mathcal{S}(x,y,\frac{1}{2})) \preceq \frac{1}{2}d(y,\mathcal{S}(x,y,\frac{1}{2})).$$

Similarly,

$$\frac{1}{2}d(y,\mathcal{S}(x,y,\frac{1}{2})) \preceq \frac{1}{2}d(x,\mathcal{S}(x,y,\frac{1}{2})).$$

Then, $d(x,\mathcal{S}(x,y,\frac{1}{2})) = d(y,\mathcal{S}(x,y,\frac{1}{2})) = \frac{1}{2}d(x,y),\ \forall x,y \in \mathcal{V}$. □

Theorem 2.3. *Let $(X,\mathbb{A},d,\mathcal{S})$ be a convex C^*-algebra valued complete metric space and $\mathcal{V}$ be a nomempty closed convex subset of X. Suppose the mapping $T:\mathcal{V}\to\mathcal{V}$ satisfies:*

$$\alpha d(x,Tx) + \beta d(y,Ty) + \gamma d(Tx,Ty) \preceq kd(x,y),\ \forall x,y \in \mathcal{V} \tag{2.1}$$

with

$$2\beta - |\gamma| \le k < 2(\alpha+\beta+\gamma) - |\gamma|.$$

Then T has least one fixed point.

Proof. Let $x_0 \in \mathcal{V}$ and define a sequence $\{x_n\}$ by

$$x_n = \mathcal{S}(x_{n-1}, Tx_{n-1}, \frac{1}{2}),\quad \forall n = 1,2,... \tag{2.2}$$

By corollary 2.2 and 2.1 we have for all $n \in \mathbb{N}$

$$d(x_n,Tx_n) = 2d(x_n,x_{n+1}) \tag{2.3}$$

$$d(x_n,Tx_{n-1}) = d(x_n,x_{n-1}) \tag{2.4}$$

we get

$$\alpha d(x_n,Tx_n) + \beta d(x_{n-1},Tx_{n-1}) + \gamma d(Tx_n,Tx_{n-1}) \preceq kd(x_n,x_{n-1}) \tag{2.5}$$

using 2.3 and 2.4 we obtain

$$2\alpha d(x_n,x_{n+1}) + 2\beta d(x_n,x_{n-1}) + \gamma d(Tx_n,Tx_{n-1}) \preceq kd(x_n,x_{n-1}),\ \forall n \in \mathbb{N} \tag{2.6}$$

If $\gamma > 0$ we have

$$2\gamma d(x_n,x_{n+1}) - \gamma d(x_n,x_{n-1}) \preceq \gamma d(Tx_n,Tx_{n-1}).$$

Similary, for $\gamma < 0$ we have

$$2\gamma d(x_n,x_{n+1}) + \gamma d(x_n,x_{n-1}) \preceq \gamma d(Tx_n,Tx_{n-1}),\ \forall n \in \mathbb{N}.$$

Therefore we have

$$2\gamma d(x_n, x_{n+1}) - |\gamma| d(x_n, x_{n-1}) \preceq \gamma d(Tx_n, Tx_{n-1}), \ \forall n \in \mathbb{N}.$$

Then

$$2\alpha d(x_n, x_{n+1}) + 2\beta d(x_n, x_{n-1}) + 2\gamma d(x_n, x_{n+1}) - |\gamma| d(x_n, x_{n-1}) \preceq kd(x_n, x_{n-1}), \ \forall n \in \mathbb{N}. \tag{2.7}$$

$$\Rightarrow d(x_n, x_{n+1}) \preceq \frac{k - 2\beta + |\gamma|}{2(\alpha+\gamma)} d(x_n, x_{n-1}), \ \forall n \in \mathbb{N}.$$

Since $\frac{k - 2\beta + |\gamma|}{2(\alpha+\gamma)} \in [0, 1)$, then $\{x_n\}$ is a Cauchy sequence in $\mathcal{V}$, there exists $u \in \mathcal{V}$ such that $\lim_{n\to\infty} x_n = u$.

By the triangle inequality, we obtain

$$\lim_{n\to\infty} Tx_n = u$$

and

$$\alpha d(u, Tu) + \beta d(x_n, Tx_n) + \gamma d(Tu, Tx_n) \preceq kd(u, x_n), \ \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, we get $(\alpha + \gamma)d(u, Tu) \preceq \theta \Rightarrow Tu = u$. □

If we put $\gamma = 0$ in Theorem 2.3 we obtain the following corollary.

Corollary 2.4. *Let $\mathcal{V}$ be a nomempty closed convex subset of a convex C^*-algebra valued complete metric space $(X, \mathbb{A}, d, \mathcal{S})$ and $T : \mathcal{V} \to \mathcal{V}$ a mapping satisfying :*

$$\alpha d(x, Tx) + \beta d(y, Ty) \preceq kd(x, y), \quad \forall x, y \in \mathcal{V}$$

with

$$2\beta \leq k < 2(\alpha + \beta).$$

Then T has least one fixed point.

Remark 2.5. If we take $\alpha = \beta = 1$ in corollary 2.4 then we obtain the following result which is a generalization of Karapinar's results.

Corollary 2.6. *Let $\mathcal{V}$ be a nomempty closed convex subset of a convex C^*-algebra valued complete metric space $(X, \mathbb{A}, d, \mathcal{S})$ and $T : \mathcal{V} \to \mathcal{V}$ a mapping satisfying*

$$\exists k \in [2, 4[, \ \forall x, y \in \mathcal{V} \ d(x, Tx) + d(y, Ty) \preceq kd(x, y). \tag{2.8}$$

Then T has at least one fixed point.

If we take $\alpha = \beta = \gamma = 1$ in Theorem 2.3, we obtain the following result.

Corollary 2.7. *Let $\mathcal{V}$ be a nomempty closed convex subset of a convex C^*-algebra valued complete metric space $(X, \mathbb{A}, d, \mathcal{S})$ and $T : \mathcal{V} \to \mathcal{V}$ a mapping satisfying*

$$\exists k \in [2, 5[, \ \forall x, y \in \mathcal{V}, \ d(x, Tx) + d(y, Ty) + d(Tx, Ty) \preceq kd(x, y). \tag{2.9}$$

Then T has at least one fixed point.

Theorem 2.8. *Let $\mathcal{V}$ be a nomempty closed convex subset of a convex C^*-algebra valued complete metric space $(X, \mathbb{A}, d, \mathcal{S})$ and $T : \mathcal{V} \to \mathcal{V}$ a mapping satisfying: $\forall x, y \in \mathcal{V},\ k \in [0, 1)$.*

$$d(Tx, Ty) \preceq k\ max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{2.10}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in \mathcal{V}$ and define a sequence $\{x_n\}$ by

$$x_n = \mathcal{S}(x_{n-1}, Tx_{n-1}, \lambda_{n-1}),\ \ and\ \lambda_{n-1} \in [0, 1)\ \forall n = 1, 2, ... \tag{2.11}$$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, W(x_n, Tx_n, \lambda_n)) \\ &\preceq (1 - \lambda_n) d(x_n, Tx_n) \end{aligned}$$

and

$$\begin{aligned} d(x_n, Tx_n) &\preceq d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) \\ &\preceq d(\mathcal{S}(x_{n-1}, Tx_{n-1}, \lambda_{n-1}), Tx_{n-1}) + kmax\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), \\ &d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &\preceq \lambda_{n-1} d(Tx_{n-1}, x_{n-1}) + kmax\{(1 - \lambda_{n-1}) d(x_{n-1}, Tx_{n-1}), \\ &d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &\preceq \lambda_{n-1} d(Tx_{n-1}, x_{n-1}) + kmax\{d(x_{n-1}, Tx_{n-1}), \\ &d(x_n, Tx_n), d(x_{n-1}, x_n) + d(x_n, Tx_n), d(\mathcal{S}(x_{n-1}, Tx_{n-1}, \lambda_{n-1}), Tx_{n-1})\} \\ &\preceq \lambda_{n-1} d(Tx_{n-1}, x_{n-1}) + kmax\{d(x_{n-1}, Tx_{n-1}), \\ &d(x_n, Tx_n), (1 - \lambda_{n-1}) d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n), \lambda_{n-1} d(x_{n-1}, Tx_{n-1})\} \\ &\preceq \lambda_{n-1} d(Tx_{n-1}, x_{n-1}) + kmax\{d(x_{n-1}, Tx_{n-1}), \\ &(1 - \lambda_{n-1}) d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)\} \\ &\preceq \lambda_{n-1} d(Tx_{n-1}, x_{n-1}) + kmax\{d(x_{n-1}, Tx_{n-1}), \\ &d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)\} \\ &= (\lambda_{n-1} + k) d(x_{n-1}, Tx_{n-1}) + kd(x_n, Tx_n). \end{aligned}$$

$$\Rightarrow d(x_n, Tx_n) \preceq \frac{\lambda_{n-1} + k}{1 - k} d(x_{n-1}, Tx_{n-1}) \tag{2.12}$$

$\Rightarrow \{d(x_n, x_{n-1})\}$ is a decreasing non negative real numbers sequence.

Therefore, $\exists\ t \succeq \theta$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = t$.

We claim that $t = \theta$. Suppose that $t \succ \theta$. Letting $n \to \infty$ in 2.12, we obtain $t \prec t$ which is a contradiction, hence $t = \theta$, then $\lim_{n\to\infty} d(x_n, Tx_n) = \theta$.

Now we shall show that $\{x_n\}$ is a Cauchy sequence in X.

Let $m, n \in \mathbb{N}$ such that $m > n$. Then

$$d(x_m, x_n) \preceq d(x_m, x_{n+1}) + d(x_{n+1}, x_n)$$

and

$$\begin{aligned}
d(x_m, x_{n+1}) &= d(\mathcal{S}(x_{m-1}, Tx_{m-1}, \lambda_{m-1}), x_{n+1}) \\
&\preceq \lambda_{m-1} d(x_{m-1}, x_{n+1}) + (1-\lambda_{m-1}) d(Tx_{m-1}, x_{n+1}) \\
&\preceq \lambda_{m-1} d(x_{m-1}, x_{n+1}) + (1-\lambda_{m-1})[d(Tx_{m-1}, Tx_{n+1}) + d(Tx_{n+1},, x_{n+1})] \\
&\preceq \lambda_{m-1} d(x_{m-1}, x_{n+1}) + (1-\lambda_{m-1}) kmax\{d(x_{m-1}, x_{n+1}), d(x_{m-1}, Tx_{m-1}), \\
&\quad d(x_{n+1}, Tx_{n+1}), d(x_{m-1}, Tx_{n+1}), d(x_{n+1}, Tx_{m-1})\} + (1-\lambda_{m-1}) d(Tx_{n+1}, x_{n+1}) \\
&\preceq \lambda_{m-1}[d(x_{m-1}, x_n) + d(x_n, x_{n+1})] + (1-\lambda_{m-1}) kmax\{d(x_{m-1}, x_{n+1}), \\
&\quad d(x_{m-1}, Tx_{m-1}), d(x_{n+1}, Tx_{n+1}), d(x_{m-1}, Tx_{n+1}), d(x_{n+1}, Tx_{m-1})\} + \\
&\quad (1-\lambda_{m-1}) d(Tx_{n+1}, x_{n+1}) \\
&\preceq \lambda_{m-1}[d(x_{m-1}, x_n) + d(x_n, x_{n+1})] + (1-\lambda_{m-1}) kmax\{d(x_{m-1}, x_n) + \\
&\quad d(x_n, x_{n+1}), d(x_{m-1}, Tx_{m-1}), d(x_{n+1}, Tx_{n+1}), d(x_{m-1}, Tx_{n+1}), d(x_{n+1}, x_{m-1}) \\
&\quad + d(x_{m-1}, Tx_{m-1})\} + (1-\lambda_{m-1}) d(Tx_{n+1}, x_{n+1}) \\
&\preceq \lambda_{m-1}[d(x_{m-1}, x_n) + d(x_n, x_{n+1})] + (1-\lambda_{m-1}) kmax\{d(x_{n+1}, Tx_{n+1}), \\
&\quad d(x_{n+1}, x_{m-1}) + d(x_{m-1}, Tx_{m-1})\} + (1-\lambda_{m-1}) d(Tx_{n+1}, x_{n+1}) \to \theta, \ (n, \ m \to \infty).
\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in $\mathcal{V}$, and by completeness of $\mathcal{V}$, $\exists u \in \mathcal{V}$ such that $\lim_{n\to\infty} d(x_n, u) = \theta$.

Now we verify that u is a fixed point of T.

$$\begin{aligned}
d(u, Tu) &\preceq d(u, x_n) + d(x_n, Tu) \\
&\preceq d(u, x_n) + d(x_n, Tx_n) + d(Tx_n, Tu) \\
&\preceq d(u, x_n) + d(x_n, Tx_n) + kmax\{d(x_n, u), d(x_n, Tx_n), d(u, Tu), d(u, Tx_n), d(x_n, Tu)\} \\
&\preceq d(u, x_n) + d(x_n, Tx_n) + kmax\{d(x_n, u), d(x_n, Tx_n), d(u, Tu), d(u, x_n) + d(x_n, Tx_n), \\
&\quad d(x_n, Tx_n) + d(Tx_n, Tu)\} \\
&\preceq d(u, x_n) + d(x_n, Tx_n) + kmax\{d(u, Tu), d(u, x_n) + d(x_n, Tx_n), \\
&\quad d(x_n, Tx_n) + d(Tx_n, Tu)\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$d(u, Tu) \preceq kd(u, Tu) \prec d(u, Tu) \Rightarrow u = Tu.$$

Hence u is a fixed point of T.

If $v \in X$ such that $Tv = v$, then

$$\begin{aligned}
d(u, v) = d(Tu, Tv) &\preceq k \ max\{d(u, v), du, Tu), d(v, Tv), d(u, Tv), d(v, Tu)\} \\
&\preceq kd(u, v) \\
&\prec d(u, v)
\end{aligned}$$

that is a contradiction. Hence, $u = v$.

□

Following is corresponding result for Chatterjea type contraction in convex C^*- algebra valued metric space.

Theorem 2.9. *Let $(X, \mathbb{A}, d, \mathcal{S})$ be a convex C^*-algebra valued complete metric space. A self map $T : X \to X$ satisfies:*

$$d(Tx, Ty) \preceq k[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$ and $k \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Proof. Let $x_0 \in \mathcal{V}$ and define a sequence $\{x_n\}$ by

$$x_n = \mathcal{S}(x_{n-1}, Tx_{n-1}, \lambda_{n-1}), \quad \text{and } \lambda_{n-1} \in [0, 1) \ \forall n = 1, 2, ... \tag{2.13}$$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, \mathcal{S}(x_n, Tx_n, \lambda_n)) \\ &\preceq (1 - \lambda_n)d(x_n, Tx_n) \end{aligned}$$

and

$$\begin{aligned} d(x_n, Tx_n) &\preceq d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) \\ &\preceq d(\mathcal{S}(x_{n-1}, Tx_{n-1}, \lambda_{n-1}), Tx_{n-1}) + k[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \\ &\preceq \lambda_{n-1}d(Tx_{n-1}, x_{n-1}) + kd(x_{n-1}, x_n) + kd(x_n, Tx_n) + kd(x_n, x_{n-1}) \\ &+ kd(x_{n-1}, Tx_{n-1}) \\ &\preceq (k + \lambda_{n-1})d(x_{n-1}, Tx_{n-1}) + 2kd(x_{n-1}, x_n) + kd(x_n, Tx_n) \end{aligned}$$

$$\Rightarrow d(x_n, Tx_n) \preceq \frac{\lambda_{n-1} + k}{1 - k}d(x_{n-1}, Tx_{n-1}) + \frac{2k}{1 - k}d(x_{n-1}, x_n) \tag{2.14}$$

$$\preceq \frac{\lambda_{n-1} + k}{1 - k}d(x_{n-1}, Tx_{n-1}) + \frac{2k}{1 - k}(1 - \lambda_{n-1})d(x_{n-1}, Tx_{n-1}) \tag{2.15}$$

$$\preceq \frac{3k - \lambda_{n-1}}{1 - k}d(x_{n-1}, Tx_{n-1}) \tag{2.16}$$

for all $n \in \mathbb{N}$, $\frac{3k - \lambda_{n-1}}{1 - k} \in [0, 1)$, and hence $\{x_n\}$ is a contraction sequence in $\mathcal{V}$. Therefore, it is a Cauchy sequence. Since $\mathcal{V}$ is a closed subset of a complete space, there $\exists u \in \mathcal{V}$ such that $lim_{n\to\infty}d(x_n, u) = \theta$.

Now we verify that u is a fixed point of T,

$$\begin{aligned} d(u, Tu) &\preceq d(u, x_n) + d(x_n, Tu) \\ &\preceq d(u, x_n) + d(x_n, Tx_n) + d(Tx_n, Tu) \\ &\preceq d(u, x_n) + d(x_n, Tx_n) + k[d(x_n, Tu) + d(u, Tx_n)] \\ &\preceq d(u, x_n) + d(x_n, Tx_n) + k[d(x_n, u) + d(u, Tu) + +d(u, x_n) + d(x_n, Tx_n)]. \end{aligned}$$

Letting $n \to \infty$, we get

$$d(u, Tu) \preceq kd(u, Tu) \prec d(u, Tu) \Rightarrow u = Tu.$$

Hence u is a fixed point of T.

Now if $x \neq y$ is another fixed point of T, then

$$d(x, y) \preceq k[d(x, y) + d(y, x)] \preceq 2kd(x, y)$$

since $k \in [0, \frac{1}{2})$, then $d(x,y) \prec d(x,y)$.

Hence $x = y$. Therefore, the fixed point is unique and the proof is complete. □

Theorem 2.10. *Let $(X, \mathbb{A}, d, \mathcal{S})$ be a convex C^*-algebra valued complete metric space and $\mathcal{V}$ be a nonempty closed convex subset of X. Suppose $T, R : \mathcal{V} \to \mathcal{V}$ are self-mappings of $\mathcal{V}$ and there exist $\alpha,\ \beta,\ \gamma,\ k$ such that:*

$$\alpha d(Rx, Tx) + \beta d(Ry, Ty) + \gamma d(Tx, Ty) \preceq kd(Rx, Ry),\ \forall x, y \in \mathcal{V} \tag{2.17}$$

with

$$2\beta - |\gamma| \leq k < 2(\alpha + \beta + \gamma) - |\gamma|. \tag{2.18}$$

If (T, R) is a Banach operator pair, R has the property (I) and $Fix(R)$ is a nonemty closed subset of $\mathcal{V}$, then $Fix(T, R)$ is nonemty.

Proof. From 2.17, we obtain

$$\alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(Tx, Ty) \preceq kd(x, y) \tag{2.19}$$

for all $x, y \in Fix(R)$.

$Fix(R)$ is convex because R has the property (I). From Theorem 2.3, we obtain that $Fix(T, R)$ is nonempty. □

3. Acknowledgments

It is our great pleasure to thank the referee for his careful reading of the paper and for several helpful suggestions.

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