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Summation of Certain Infinite Series by the Approach of Partial Fractions and Hypergeometric Functions

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ABSTRACT. In the present article, we have obtained the summations of certain infinite series by using partial fraction approach, some hypergeometric summation theorems of positive and negative unit arguments, Digamma, trigamma, tetragamma, Riemann Zeta functions, lower case beta function of one-variable and other associated functions. We have also obtained some new hypergeometric summation theorems of positive and negative unit arguments.

1. Introduction and preliminaries

The enormous popularity and broad usefulness of the hypergeometric function ${}_2F_1$ and the generalized hypergeometric functions ${}_pF_q$ $(p, q \in \mathbb{N}_0)$ of one variable have inspired and stimulated a large number of researchers to introduce and investigate hypergeometric functions of two or more variables (see, e.g., [4]). A serious, significant, and systematic study of the hypergeometric functions of two variables was initiated by Appell [3], who offered the so-called Appell functions F_1, F_2, F_3, and F_4 which are generalizations of the Gauss hypergeometric function. The confluent forms of the Appell functions were studied by Humbert [8]. A complete list of these functions can be seen in the standard literature. Later, the four Appell functions and their confluent forms were further generalized by Kampé de Fériet, who introduced more general hypergeometric functions of two variables. The notation defined and introduced by Kampé de Fériet for his double-hypergeometric functions of superior order was subsequently abbreviated by Burchnall and Chaundy [5].

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$ {}_pF_q\left[\begin{array}{c} (\alpha_p); \\ \\ (\beta_q); \end{array} z\right] = {}_pF_q\left[\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n} \frac{z^n}{n!}, \tag{1.1} $$

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is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \ldots \; ; \; j = 1, 2, \ldots, q. \tag{1.2}$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,
(ii) converges for $|z| < 1$, if $p = q + 1$,
(iii) diverges for all z, $z \neq 0$, if $p > q + 1$,
(iv) converges absolutely for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) > 0$,
(v) converges conditionally for $|z| = 1$ $(z \neq 1)$, if $p = q + 1$ and $-1 < \Re(\omega) \leqq 0$,
(vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) \leqq -1$,

where, by convention, a product over an empty set is interpreted as 1 and

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j = \tag{1.3}$$

= Sum of denominator parameters − Sum of numerator parameters.

In this paper, we shall use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \cdots\}\,; \mathbb{N}_0 := \mathbb{N} \bigcup \{0\}\,; \mathbb{Z}_0^- := \mathbb{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}$.
The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.
The Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ [12, p.22, Eq.(1), p.32, Q.N.(8) and Q.N.(9), see also [15] p.23, Eq.(22) and Eq.(23)] is defined by:

$$(\alpha)_p := \frac{\Gamma(\alpha + p)}{\Gamma(\alpha)} = \begin{cases} 1 & ; (p = 0; \alpha \in \mathbb{C} \backslash \{0\}), \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & ; (p = n \in \mathbb{N}; \alpha \in \mathbb{C}), \\ \frac{(-1)^k n!}{(n-k)!} & ; (\alpha = -n; p = k; n, k \in \mathbb{N}_0; 0 \leq k \leq n), \\ 0 & ; (\alpha = -n; p = k; n, k \in \mathbb{N}_0; k > n), \\ \frac{(-1)^k}{(1-\alpha)_k} & ; (p = -k; k \in \mathbb{N}; \alpha \in \mathbb{C} \backslash \mathbb{Z}), \end{cases}$$

it being understood conventionally that$(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists (see, for details, [15, p.21 *et seq.*]).
The Riemann Zeta function $\zeta(z)$ [10, p.19] is defined as:

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}; \; \mathfrak{Re}(z) > 1. \tag{1.4}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^z} = (2^{1-z} - 1)\zeta(z); \; \mathfrak{Re}(z) > 0. \tag{1.5}$$

The Catalan constant is defined as:

$$\mathbf{G} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = {}_3F_2\left[\begin{array}{c} 1,\ \frac{1}{2},\ \frac{1}{2}; \\ \\ \frac{3}{2},\ \frac{3}{2}; \end{array} -1\right] = 0.9159655942... \tag{1.6}$$

The logarithmic derivative of the Gamma function also known as psi function or Digamma function [12, p.10, Eq.(1), [14], p.24, Eq.(2), [9], p.12, Eq.(1)], is defined as:

$$\psi(z) = \frac{d}{dz}\ell n\left\{\Gamma(z)\right\} = \frac{\Gamma'(z)}{\Gamma(z)};\ z \neq 0, -1, -2, -3, ... \tag{1.7}$$

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)};\ z \neq 0, -1, -2, -3, ..., \tag{1.8}$$

$$\psi(z) = -\gamma - \sum_{n=0}^{\infty} \left\{\frac{1}{(z+n)} - \frac{1}{(n+1)}\right\};\ z \neq 0, -1, -2, -3, ..., \tag{1.9}$$

where γ is Euler-Mascheroni constant and $\gamma \cong 0.577215664901532860606512....$

$$\psi(1) = -\gamma,\ \psi\left(\frac{1}{2}\right) = -2\ \ell n\ 2 - \gamma,\ \psi\left(\frac{3}{2}\right) = 2 - 2\ \ell n\ 2 - \gamma, \tag{1.10}$$

$$\psi\left(\frac{7}{6}\right) = 6 - \gamma - \frac{\pi\sqrt{3}}{2} - \frac{3}{2}\ \ell n\ 3 - 2\ \ell n\ 2,\ \psi\left(\frac{5}{6}\right) = -\gamma + \frac{\pi\ \sqrt{3}}{2} - \frac{3}{2}\ell n\ 3 - 2\ell n\ 2. \tag{1.11}$$

$$\psi\left(\frac{5}{2}\right) = \frac{8}{3} - 2\ell n 2 - \gamma,\ \psi\left(\frac{1}{2}\right) = -2\ \ell n\ 2 - \gamma, \tag{1.12}$$

$$\psi^{(2)}\left(\frac{1}{2}\right) = -\frac{14\pi^3}{25.79436},\ \psi^{(2)}\left(\frac{5}{2}\right) = -\frac{14\pi^3}{25.79436} + \frac{448}{27}, \tag{1.13}$$

$$\psi^{(1)}\left(\frac{1}{2}\right) = \frac{\pi^2}{2},\ \psi^{(1)}\left(\frac{3}{2}\right) = \frac{\pi^2}{2} - 4,\ \psi^{(1)}\left(\frac{5}{2}\right) = \frac{\pi^2}{2} - 4.4. \tag{1.14}$$

The polygamma function $\psi^{(n)}(z)$ ([14, p.33, Eq.(52), Eq.(53), p.34, Eq.(58)], see also [1, p.260, Eq.(6.4.10), Eq.(6.4.4), [7], p.45, Eq.(9), [10], p.15]), is defined as:

$$\psi^{(n)}(z) = \frac{d^{n+1}}{d\ z^{n+1}}\ell n\left(\Gamma(z)\right) = \frac{d^n}{d\ z^n}\psi(z);\ n \in \mathbb{N}_0,\ z \neq 0, -1, -2, ... \tag{1.15}$$

$$\psi^{(n)}(z) = (-1)^{n+1}\ n!\ \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}};\ n \in \mathbb{N},\ z \neq 0, -1, -2, ... \tag{1.16}$$

Lower case beta function of one variable:

$$\beta(z) = \frac{1}{2}\left[\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right)\right] = \frac{G(z)}{2},\ z \neq 0, -1, -2, -3, ... \tag{1.17}$$

$$\beta(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)} = \frac{1}{z}\ {}_2F_1\left[\begin{array}{c} 1,\ z; \\ \\ 1+z; \end{array} -1\right],\ -z \neq 0, 1, 2, 3, ... \tag{1.18}$$

$$\beta^{(n)}(z) = \frac{d^n}{dz^n}\beta(z) = (-1)^n\ n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)^{n+1}};\ -z \in \mathbb{N}_0. \tag{1.19}$$

$$\beta(1) = \ell n\ 2,\ \beta(2) = 1 - \ell n\ 2,\ \beta^{(1)}(1) = -\frac{\pi^2}{12}, \tag{1.20}$$

$$\beta^{(1)}\left(\frac{3}{2}\right) = 4\mathbf{G} - 4,\ \beta^{(1)}(2) = \frac{\pi^2}{12} - 1, \tag{1.21}$$

$$\beta\left(\frac{1}{2}\right) = \frac{\pi}{2},\ \beta\left(\frac{1}{4}\right) = \frac{\pi\sqrt{2}}{2} + \sqrt{2}\ \ell n(\sqrt{2}+1),\ \beta\left(\frac{3}{2}\right) = \frac{4-\pi}{2}. \tag{1.22}$$

Some hypergeometric summation theorems in terms of Digamma $\psi(b)$, trigamma $\psi^{(1)}(b)$, tetragamma $\psi^{(2)}(b)$ functions and derivatives of lower case Beta function of one-variable are given below:
See ref. [11, p. 489, Entry (7.3.6.(9))]

$${}_2F_1\left[\begin{array}{c} 1,\ a; \\ \\ a+1; \end{array} -1\right] = a\beta(a);\ 1+a \in \mathbb{C}\backslash\mathbb{Z}_0^-. \tag{1.23}$$

See ref. [11, p. 536, Entry (7.4.4.(34))]

$${}_3F_2\left[\begin{array}{c} 1,\ b,\ b; \\ \\ b+1,\ b+1; \end{array} 1\right] = b^2\ \psi^{(1)}(b), \tag{1.24}$$

where $1+b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $b = a$.
See ref. [11, p. 546, Entry (7.4.5.(5))]

$${}_3F_2\left[\begin{array}{c} 1,\ a,\ a; \\ \\ a+1,\ a+1; \end{array} -1\right] = -a^2\ \beta^{(1)}(a), \tag{1.25}$$

where $1+a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $b = a$.

See ref. [11, p. 554, Entry (7.5.3.(3))]

$${}_4F_3\left[\begin{array}{c} 1,\ a,\ b,\ c; \\ \\ 1+a, 1+b, 1+c; \end{array} 1\right] = -abc\left[\frac{\psi(a)}{(b-a)(c-a)} + \frac{\psi(b)}{(a-b)(c-b)} + \frac{\psi(c)}{(a-c)(b-c)}\right], \tag{1.26}$$

where $1+a,\ 1+b,\ 1+c \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $a \neq b, b \neq c, a \neq c$.
See ref. [11, p. 554, Entry (7.5.3.(5))]

$${}_4F_3\left[\begin{array}{c} 1,\ b,\ b,\ b; \\ \\ b+1,\ b+1,\ b+1; \end{array} 1\right] = \frac{-b^3}{2}\ \psi^{(2)}(b), \tag{1.27}$$

where $1+b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $a=b=c$.
Dougall's theorem [2, p.71, Eq.(2.2.10), p.147, Entry(3.5.2), [6], [11], p.564, Entry(7.6.2(3)) [13], p.56, Eq.(2.3.4.5), p.244, Entry(III.12), see also [4], p.27, Eq.(4.4(1))] in terms of Gamma function is given as:

$$ {}_5F_4\left[\begin{array}{c} a,\ 1+\frac{a}{2},\ b,\ c,\ d; \\ \qquad\qquad\qquad\qquad 1 \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d; \end{array}\right] $$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}, \tag{1.28} $$

provided $\mathfrak{Re}(a-b-c-d) > -1$ and $\frac{a}{2},\ 1+a-b,\ 1+a-c,\ 1+a-d \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

The present article is organized as follows. In section 2, we have shown that the difference of two divergent series may be convergent. In Section 3, we have obtained the summations of certain infinite series whose general terms are rational functions of n, by using partial fraction and some hypergeometric summation theorems of positive and negative unit arguments. In section 4, we have obtained some new hypergeometric summation theorems.

2. Difference of two divergent Gauss' series

Example : Consider the two positive terms infinite series $\sum_{n=0}^{\infty}\frac{1}{(1+2n)}$ and $\sum_{n=0}^{\infty}\frac{1}{(5+2n)}$, which are divergent in nature by using the comparison test.
Taking the difference of the above two series, we get

$$ \sum_{n=0}^{\infty}\frac{1}{(1+2n)} - \sum_{n=0}^{\infty}\frac{1}{(5+2n)} = \sum_{n=0}^{\infty}\frac{4}{(1+2n)(5+2n)}. \tag{2.1} $$

The right hand side of equation (2.1) is convergent by using the Raabe's higher ratio test. In terms of hypergeometric function, the equation (2.1) can be written as

$$ \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} - \frac{1}{5}\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n} = \frac{4}{5}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{5}{2}\right)_n}{\left(\frac{3}{2}\right)_n\left(\frac{7}{2}\right)_n}, $$

$$ {}_2F_1\left[\begin{array}{c} \frac{1}{2},\ 1; \\ \qquad 1 \\ \frac{3}{2}; \end{array}\right] - \frac{1}{5}\,{}_2F_1\left[\begin{array}{c} \frac{5}{2},\ 1; \\ \qquad 1 \\ \frac{7}{2}; \end{array}\right] = \frac{4}{5}\,{}_3F_2\left[\begin{array}{c} \frac{1}{2},\ \frac{5}{2},\ 1; \\ \qquad 1 \\ \frac{3}{2},\ \frac{7}{2}; \end{array}\right]. \tag{2.2} $$

Since both the Gauss' series having the positive unit argument on left hand side of equation (2.2) are divergent. But their difference is convergent.
Multiplying both sides of equation (2.2) by $\frac{15}{2048}$, for application point of view in next section, we get

Corollary 1. *The difference of two divergent Gauss' series having the positive unit argument may be convergent*

$$\frac{15}{2048}\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\ 1;\\ & 1\\ \frac{3}{2};\end{matrix}\right]-\frac{3}{2048}\,{}_2F_1\left[\begin{matrix}\frac{5}{2},\ 1;\\ & 1\\ \frac{7}{2};\end{matrix}\right]=\frac{3}{512}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{5}{2},\ 1;\\ & 1\\ \frac{3}{2},\ \frac{7}{2};\end{matrix}\right]. \tag{2.3}$$

3. Summation of some infinite series

1. $$\sum_{n=0}^{\infty}\frac{1}{(256n^8+3072n^7+15360n^6+41472n^5+65568n^4+61632n^3+33440n^2+9600n+1125)}=\frac{1}{27}-\frac{15\pi^2}{4096}. \tag{3.1}$$

2. $$\sum_{n=0}^{\infty}\frac{(32n^4+120n^3+156n^2+82n+15)\left\{\left(\frac{1}{2}\right)_n\right\}^2\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(n!)^2(1296n^5+9072n^4+24552n^3+32112n^2+20341n+5005)\left(\frac{5}{6}\right)_n\left(\frac{7}{6}\right)_n}=\frac{1}{54\pi\sqrt{3}}. \tag{3.2}$$

3. $$\sum_{n=0}^{\infty}\frac{(5+8n)\left(\frac{1}{4}\right)_{n+1}\left\{\left(\frac{1}{4}\right)_n\right\}^3}{(n!)^4\,(n^3+3n^2+3n+1)}=\frac{64\sqrt{2}}{27\sqrt{\pi}\left[\Gamma\left(\frac{3}{4}\right)\right]^2}. \tag{3.3}$$

4. $$\sum_{n=0}^{\infty}\frac{(8n^2+10n+3)\ \left\{\left(\frac{1}{2}\right)_n\right\}^2\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(n!)^2\,(216n^4+684n^3+750n^2+317n+35)\left(\frac{1}{6}\right)_n\left(\frac{5}{6}\right)_n}=\frac{1}{2\pi\sqrt{3}}. \tag{3.4}$$

5. $$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(4n^6+36n^5+133n^4+258n^3+277n^2+156n+36)}=12\ell n\,2-23+16\mathbf{G}. \tag{3.5}$$

6. $$\sum_{n=0}^{\infty}\frac{1}{(8n^3+24n^2+22n+6)}=\ell n\,2-\frac{1}{2}. \tag{3.6}$$

7. $$\sum_{n=0}^{\infty}\frac{1}{(36n^3+108n^2+107n+35)}=-3+\frac{3}{2}\,\ell n\,3+2\,\ell n\,2. \tag{3.7}$$

8. $$\sum_{n=0}^{\infty}\frac{(1+n)^2}{(16n^4+64n^3+88n^2+48n+9)}=\frac{\pi^2}{64}. \tag{3.8}$$

9. $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(16n^4+52n^3+56n^2+23n+3)}=\frac{1}{3}\,\ell n\,2+\frac{\pi}{5}\left(\frac{2\sqrt{2}}{3}-1\right)+\frac{4\sqrt{2}}{15}\,\ell n\,(1+\sqrt{2})-\frac{1}{5}. \tag{3.9}$$

Proof of the result (3.1) :

On factorizing the general term of equation (3.1) and making use of partial fractions, we have

$$\frac{1}{(256n^8+3072n^7+15360n^6+41472n^5+65568n^4+61632n^3+33440n^2+9600n+1125)}$$
$$=\frac{\frac{15}{2048}}{(2n+1)}+\frac{\frac{-7}{1024}}{(2n+1)^2}+\frac{\frac{1}{256}}{(2n+1)^3}+\frac{\frac{-1}{64}}{(2n+3)^2}+\frac{\frac{-15}{2048}}{(2n+5)}+\frac{\frac{-7}{1024}}{(2n+5)^2}+\frac{\frac{-1}{256}}{(2n+5)^3}.$$

Now taking summation on both sides and n varying from 0 to ∞, we get

$$\sum_{n=0}^{\infty}\frac{1}{(256n^8+3072n^7+15360n^6+41472n^5+65568n^4+61632n^3+33440n^2+9600n+1125)}$$

$$=\frac{15}{2048}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n}-\frac{7}{1024}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}+\frac{1}{256}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}-\frac{1}{576}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n\left(\frac{5}{2}\right)_n}-$$
$$-\frac{3}{2048}\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n}-\frac{7}{25600}\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2}\right)_n\left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n\left(\frac{7}{2}\right)_n}-\frac{1}{32000}\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2}\right)_n\left(\frac{5}{2}\right)_n\left(\frac{5}{2}\right)_n}{\left(\frac{7}{2}\right)_n\left(\frac{7}{2}\right)_n\left(\frac{7}{2}\right)_n}.$$

Using the definition of generalized hypergeometric function, we get

$$\sum_{n=0}^{\infty}\frac{1}{(256n^8+3072n^7+15360n^6+41472n^5+65568n^4+61632n^3+33440n^2+9600n+1125)}$$

$$=\frac{15}{2048}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ 1;\\ \\ \frac{3}{2};\end{array}\ 1\right]-\frac{7}{1024}\,{}_3F_2\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2},\ 1;\\ \\ \frac{3}{2},\ \frac{3}{2};\end{array}\ 1\right]+\frac{1}{256}\,{}_4F_3\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ 1;\\ \\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2};\end{array}\ 1\right]-$$
$$-\frac{1}{576}\,{}_3F_2\left[\begin{array}{c}\frac{3}{2},\ \frac{3}{2},\ 1;\\ \\ \frac{5}{2},\ \frac{5}{2};\end{array}\ 1\right]-\frac{3}{2048}\,{}_2F_1\left[\begin{array}{c}\frac{5}{2},\ 1;\\ \\ \frac{7}{2};\end{array}\ 1\right]-$$
$$-\frac{7}{25600}\,{}_3F_2\left[\begin{array}{c}\frac{5}{2},\ \frac{5}{2},\ 1;\\ \\ \frac{7}{2},\ \frac{7}{2};\end{array}\ 1\right]-\frac{1}{32000}\,{}_4F_3\left[\begin{array}{c}\frac{5}{2},\ \frac{5}{2},\ \frac{5}{2},\ 1;\\ \\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2};\end{array}\ 1\right]. \tag{3.10}$$

Using summation theorems (1.24), (1.27) and corollary (2.3) in equation (3.10), we get

$$\sum_{n=0}^{\infty}\frac{1}{(256n^8+3072n^7+15360n^6+41472n^5+65568n^4+61632n^3+33440n^2+9600n+1125)}$$

$$=\frac{15}{4096}\left\{\psi\left(\frac{5}{2}\right)-\psi\left(\frac{1}{2}\right)\right\}-\frac{7}{4096}\,\psi^{(1)}\left(\frac{1}{2}\right)-\frac{1}{4096}\,\psi^{(2)}\left(\frac{1}{2}\right)-$$
$$-\frac{9}{2304}\,\psi^{(1)}\left(\frac{3}{2}\right)-\frac{175}{102400}\,\psi^{(1)}\left(\frac{5}{2}\right)+\frac{125}{512000}\,\psi^{(2)}\left(\frac{5}{2}\right)$$
$$=\frac{15}{4096}\left\{\frac{8}{3}-2\ell n2-\gamma-(-2\,\ell n2-\gamma)\right\}-\frac{7}{4096}\left(\frac{\pi^2}{2}\right)-\frac{1}{4096}\left(-\frac{14\pi^3}{25.79436}\right)-$$
$$-\frac{9}{2304}\left(\frac{\pi^2}{2}-4\right)-\frac{175}{102400}\left(\frac{\pi^2}{2}-4.4\right)+\frac{125}{512000}\left(-\frac{14\pi^3}{25.79436}+\frac{448}{27}\right)$$
$$=\frac{40}{4096}+\frac{36}{2304}+\frac{7000}{921600}+\frac{448}{110592}-\pi^2\left(\frac{7}{8192}+\frac{9}{4608}+\frac{175}{204800}\right).$$

On simplifying further, we arrive at the result (3.1).

Proof of the results (3.2) to (3.9):

The proof of results (3.2) to (3.9) can be obtained by following the same procedure as in the proof of the result (3.1) and making use of Dougall's theorem (1.28) and other summation theorems. So we omit the details here.

4. Representation of infinite series (3.1) to (3.9) in hypergeometric forms

$$ {}_5F_4\left[\begin{matrix} \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{5}{2},\ 1; \\ \\ \frac{3}{2},\ \frac{7}{2},\ \frac{7}{2},\ \frac{7}{2}; \end{matrix}\ 1\right] = \frac{125}{3} - \frac{16875\pi^2}{4096}. \tag{4.1} $$

$$ {}_5F_4\left[\begin{matrix} \frac{9}{4},\ \frac{3}{2},\ \frac{5}{2},\ \frac{1}{3},\ \frac{2}{3}; \\ \\ 2,\ \frac{5}{4},\ \frac{17}{6},\ \frac{19}{6}; \end{matrix}\ 1\right] = \frac{1001}{162\pi\sqrt{3}}. \tag{4.2} $$

$$ {}_5F_4\left[\begin{matrix} \frac{13}{8},\ \frac{5}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}; \\ \\ 2,\ 2,\ 2,\ \frac{5}{8}; \end{matrix}\ 1\right] = \frac{256\sqrt{2}}{135\sqrt{\pi}\left[\Gamma\left(\frac{3}{4}\right)\right]^2}. \tag{4.3} $$

$$ {}_5F_4\left[\begin{matrix} \frac{7}{4},\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}; \\ \\ 2,\ \frac{3}{4},\ \frac{11}{6},\ \frac{13}{6}; \end{matrix}\ 1\right] = \frac{35}{6\pi\sqrt{3}}. \tag{4.4} $$

$$ {}_5F_4\left[\begin{matrix} 1\ ,\ 1,\ 1,\ \frac{3}{2},\ \frac{3}{2}; \\ \\ 3,\ 3,\ \frac{5}{2},\ \frac{5}{2}; \end{matrix}\ -1\right] = 828 - 432\ \ell n\ 2 - 576\mathbf{G}. \tag{4.5} $$

$$ {}_4F_3\left[\begin{matrix} 1,\ 1,\ \frac{1}{2},\ \frac{3}{2}; \\ \\ 2,\ \frac{3}{2},\ \frac{5}{2}; \end{matrix}\ 1\right] = 6\ \ell n\ 2 - 3. \tag{4.6} $$

$$ {}_4F_3\left[\begin{matrix} 1,\ 1,\ \frac{5}{6},\ \frac{7}{6}; \\ \\ 2,\ \frac{11}{6},\ \frac{13}{6}; \end{matrix}\ 1\right] = -105 + \frac{105}{2}\ \ell n\ 3 + 70\ \ell n\ 2. \tag{4.7} $$

$$ {}_4F_3\left[\begin{matrix} \frac{1}{2},\ \frac{1}{2},\ 2,\ 2; \\ \\ \frac{5}{2},\ \frac{5}{2},\ 1; \end{matrix}\ 1\right] = \frac{9\pi^2}{64}. \tag{4.8} $$

$$ {}_4F_3\left[\begin{matrix} 1,\ 1,\ \frac{1}{2},\ \frac{1}{4}; \\ \\ 2,\ \frac{5}{2},\ \frac{5}{4}; \end{matrix}\ -1\right] = \ell n\ 2 + \frac{3\pi}{5}\left(\frac{2\sqrt{2}}{3} - 1\right) + \frac{4\sqrt{2}}{5}\ \ell n(1+\sqrt{2}) - \frac{3}{5}. \tag{4.9} $$

Proof of the results (4.1) to (4.9): The proof of results (4.1)to (4.9) can be obtained by using the definition of generalized hypergeometric function of one variable.,

5. Conclusion

In this paper, we have obtained the summations of certain infinite series by using partial fraction and some hypergeometric summation theorems of positive and negative unit arguments, Digamma, trigamma, tetragamma functions, lower case beta function of one-variable and other associated functions. We have also obtained some hypergeometric summation theorems. We conclude this paper with the remark that the summation of other infinite series can be derived in a same way. Besides presented infinite series is supposed to find various applications in Numerical Analysis, Statistics and Linear Programming.

References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Reprint of the 1972 Edition, Dover Publications, Inc., New York, 1992.

[2] G.E. Andrews, R. Askey and R. Roy, Special Functions, Cambridge University Press, Cambridge, UK, 1999.

[3] P. Appell and J. Kampé de Fériet, Fonctions Hypergéométriques et Hypersphériques-Polynômes d' Hermite, Gauthier-Villars, Paris, 1926.

[4] W.N. Bailey, Generalized Hypergeometric Series, Cambridge University Press, London, 1935.

[5] J. L. Burchnall and T. W. Chaundy, Expansions of Appell's double hypergeometric functions (II), Quart. J. Math. Oxford Ser. 12 (1941), 112-128.

[6] J. Dougall, On Vandermonde's theorem and some more general expansions, Proc. Edinburg Math. Soc. 25 (1907), 114-132.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Higher Transcendental Functions, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, 1955.

[8] P. Humbert, The confluent hypergeometriques d' order superieur a deux variables, C. R. Acad. Sci. Paris, 173 (1921), 73-96.

[9] Y.L. Luke, The Special Functions and Their Approximations, Vol. I, Academic Press, 1969.

[10] W. Magnus, F. Oberhettinger and R.P.Soni, Some Formulas and Theorems for the Special Functions of Mathematical Physics, Third Enlarged Edition, Springer-Verlag, New York, 1966.

[11] A.P. Prudnikov, Yu. A. Brychknov and O.I. Marichev, Integrals and Series, Vol. III: More special functions, Nauka Moscow, 1986 (in Russian);(Translated from the Russian by G.G. Gould), Gordon and Breach Science Publishers, New York, Philadelphia London, Paris, Montreux, Tokyo, Melbourne, 1990.

[12] E.D. Rainville, Special Functions, The Macmillan Co. Inc., New York 1960; Reprinted by Chelsea publ. Co., Bronx, New York, 1971.

[13] L.J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, New York, 1966.

[14] H.M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier, 2011.

[15] H.M. Srivastava and H.L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.