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Controlled $*$-K-Operator Frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$

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Abstract. Frame Theory has a great revolution for recent years. This theory has been extended from Hilbert spaces to Hilbert C^*-modules. In this paper, we introduce the concept of Controlled $*$-K-operator frame for the space $End_{\mathcal{A}}^{*}(\mathcal{H})$ of all adjointable operators on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ and we establish some results.

1. Introduction and preliminaries

In 1946, Gabor [8] introduced a method for reconstructing functions (signals) using a family of elementary functions. Later in 1952, Duffin and Schaeffer [6] presented a similar tool in the context of nonharmonic Fourier series and this is the starting point of frame theory. After some decades, Daubechies, Grossmann and Meyer [4] in 1986 announced formally the definition of frame in the abstract Hilbert spaces.

Controlled frames in Hilbert spaces have been introduced by P. Balazs [2] to improve the numerical efficiency of iterative algorithms for inverting the frame operator.

In this paper, we define and study the notion of controlled $*$-K-operator frames for $End_{\mathcal{A}}^{*}(\mathcal{H})$, to consider the relation between $*$-K-operator frames and controlled $*$-K-operator frames in a given Hilbert C^*-module and we establish some results.

In the following, we briefly recall the definitions and basic properties of C^*-algebra, Hilbert $\mathcal{A}$-modules. Our references for C^*-algebras as [3, 5]. For a C^*-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of all positive elements of $\mathcal{A}$.

Definition 1.1. [9] Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.
(ii) $\langle ax + y, z \rangle_{\mathcal{A}} = a \langle x, z \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.
(iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^{*}$ for all $x, y \in \mathcal{H}$.

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For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x\rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x\rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules, A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We reserve the notation $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$ is abbreviated to $End^*_{\mathcal{A}}(\mathcal{H})$. Also, $GL^+(\mathcal{H})$ is the set of all positive bounded linear invertible operators on $\mathcal{H}$ with bounded inverse.

The following lemmas will be used to prove our mains results.

Lemma 1.2. *[1]. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:*

(i) *T is surjective.*
(ii) *T^* is bounded below with respect to norm, i.e.: there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$, $x \in \mathcal{K}$.*
(iii) *T^* is bounded below with respect to the inner product, i.e.: there is $m' > 0$ such that,*

$$\langle T^*x, T^*x\rangle_{\mathcal{A}} \geq m'\langle x, x\rangle_{\mathcal{A}}, \quad x \in \mathcal{K}.$$

Lemma 1.3. *[9]. Let $\mathcal{H}$ be an Hilbert $\mathcal{A}$-module. If $T \in End^*_{\mathcal{A}}(\mathcal{H})$, then*

$$\langle Tx, Tx\rangle_{\mathcal{A}} \leq \|T\|^2\langle x, x\rangle_{\mathcal{A}}, \qquad x \in \mathcal{H}.$$

For the following theorem, $R(T)$ denote the range of the operator T.

Theorem 1.4. *[7] Let E, F and G be Hilbert $\mathcal{A}$-modules over a C^*-algebra $\mathcal{A}$. Let $T \in End^*_{\mathcal{A}}(E, F)$ and $T' \in End^*_{\mathcal{A}}(G, F)$ with $\overline{(R(T^*))}$ is orthogonally complemented. Then the following statements are equivalent:*

(1) *$T'(T')^* \leq \lambda TT^*$ for some $\lambda > 0$.*
(2) *There exists $\mu > 0$ such that $\|(T')^*x\| \leq \mu\|T^*x\|$ for all $x \in F$.*
(3) *There exists $D \in End^*_{\mathcal{A}}(G, E)$ such that $T' = TD$, that is the equation $TX = T'$ has a solution.*
(4) *$R(T') \subseteq R(T)$.*

2. Controlled $*$-K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$

We begin this section with the following definition.

Definition 2.1. Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $C, C' \in GL^+(\mathcal{H})$. A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be a (C, C')-controlled $*$-K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$, if there exist two strictly non zero elements A and B in $\mathcal{A}$ such that

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle T_iCx, T_iC'x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}B^*, x \in \mathcal{H}. \tag{2.1}$$

The elements A and B are called respectively lower and upper bounds of the (C, C')-controlled $*$-K-operator frame.

If $A\langle K^{*}x, K^{*}x\rangle_{\mathcal{A}}A^{*} = \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}}$, the $(C, C^{'})$-controlled $*$-K-operator frame is called A-tight.
If $A = 1_{\mathcal{A}}$, it is called a normalized tight $(C, C^{'})$-controlled $*$-K-operator frame or a Parseval $(C, C^{'})$-controlled $*$-K-operator frame.

Example 2.2. Let $\mathcal{H} = l_2(\mathbb{C}) = \left\{\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C};\ \sum_{n\in\mathbb{N}}|a_n|^2 < \infty\right\}$ be a Hilbert space with respect the inner product,

$$\langle\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}\rangle = \sum_{n\in\mathbb{N}} a_n\bar{b_n},$$

equiped with the norm,

$$\|\{a_n\}_{n\in\mathbb{N}}\|_{l_2(\mathbb{C})} = (\sum_{n\in\mathbb{N}}|a_n|^2)^{\frac{1}{2}}.$$

We consider the $\mathbb{C}^*$-algebra $\mathcal{A} = \left\{\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C};\ \underset{n\in\mathbb{N}}{max}|a_n| < \infty\right\}$, equiped with the involution,

$$\{a_n\}_{n\in\mathbb{N}} \longrightarrow \{a_n\}^*_{n\in\mathbb{N}} = \{\bar{a_n}\}_{n\in\mathbb{N}} \quad for\ all \quad \{a_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$$

Let the map,

$$\mathcal{H}\times\mathcal{H} \longrightarrow \mathcal{A}$$

$$(\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}) \longrightarrow \langle\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}\rangle_{\mathcal{A}} = \left\{\frac{a_n\overline{b_n}}{n}\right\}_{n\in\mathbb{N}}$$

Wich is an inner product and equiped with it, $\mathcal{H}$ is a Hilbert $\mathcal{A}$-modules.
Let

$$\Lambda_k : \mathcal{H} \longrightarrow \mathcal{H}$$

$$\{a_n\}_{n\in\mathbb{N}} \longrightarrow \left\{\frac{a_n\delta_n^{2k+1}}{\sqrt{2k+1}}\right\}_{n\in\mathbb{N}},$$

where δ_n^{2k+1} is the Kronecker symbol.
Λ_k is defined as follow,

$$\Lambda_k(\{a_n\}_{n\in\mathbb{N}}) = (0, 0, ..., \frac{a_{2k+1}}{\sqrt{2k+1}}, 0, ...) \quad For\ all \quad \{a_n\}_{n\in\mathbb{N}} \subset \mathcal{H}.$$

Hence the family $\{\Lambda_k\}_{k\in\mathbb{N}}$ is bounded and linear.
For $\alpha, \beta > 0$, we define,

$$C : \mathcal{H} \longrightarrow \mathcal{H}$$

$$\{a_n\}_{n\in\mathbb{N}} \longrightarrow \{\alpha a_n\}_{n\in\mathbb{N}},$$

and

$$C^{'} : \mathcal{H} \longrightarrow \mathcal{H}$$

$$\{a_n\}_{n\in\mathbb{N}} \longrightarrow \{\beta a_n\}_{n\in\mathbb{N}},$$

It's easy to see that C and $C^{'}$ are bounded, linear and invertible operators. Furthermore,

$$\sum_{k\in\mathbb{N}}\langle\Lambda_k C(\{a_n\}_{n\in\mathbb{N}}),\Lambda_k C^{'}(\{a_n\}_{n\in\mathbb{N}})\rangle_{\mathcal{A}} = \sum_{k\in\mathbb{N}}\langle\Lambda_k(\{\alpha a_n\}_{n\in\mathbb{N}}),\Lambda_k(\{\beta a_n\}_{n\in\mathbb{N}})\rangle_{\mathcal{A}}$$
$$= \sum_{k\in\mathbb{N}}\langle\left\{\frac{\alpha a_n\delta_n^{2k+1}}{\sqrt{2k+1}}\right\}_{n\in\mathbb{N}},\left\{\frac{\beta a_n\delta_n^{2k+1}}{\sqrt{2k+1}}\right\}_{n\in\mathbb{N}}\rangle_{\mathcal{A}}$$
$$= \sum_{k\in\mathbb{N}}\alpha\beta\left\{\frac{|a_n|^2\delta_n^{2k+1}}{(2k+1)n}\right\}_{n\in\mathbb{N}}$$
$$= \alpha\beta\left\{\frac{|a_{2k+1}|^2}{(2k+1)^2}\right\}_{k\in\mathbb{N}}$$

Then the family $\{\Lambda_k\}_{k\in\mathbb{N}}$ is a $*$-$(C,C^{'})$-controlled Bessel sequence, but is not a $(C,C^{'})$-controlled operator frame.
Indeed,

$$\sum_{k\in\mathbb{N}}\langle\Lambda_k C\{a_n\}_{n\in\mathbb{N}},\Lambda_k C^{'}\{a_n\}_{n\in\mathbb{N}}\rangle_{\mathbb{A}} \leq \alpha\beta\left\{\frac{|a_{2k+1}|^2}{(2k+1)}\right\}_{k\in\mathbb{N}}$$
$$\leq \alpha\beta\left\{\frac{|a_k|^2}{k}\right\}_{k\in\mathbb{N}}$$
$$= \alpha\beta\langle\{a_n\}_{n\in\mathbb{N}},\{a_n\}_{n\in\mathbb{N}}\rangle_{\mathbb{A}}$$
$$= \sqrt{\alpha\beta}Id_{\mathbb{A}}\langle\{a_n\}_{n\in\mathbb{N}},\{a_n\}_{n\in\mathbb{N}}\rangle_{\mathbb{A}}\sqrt{\alpha\beta}Id_{\mathcal{A}}$$

But if $\{a_n\}_{n\in\mathbb{N}}\in\mathcal{H}$ and $a_{k+1}=0 \quad \forall k\in\mathbb{N}$, then

$$\langle\Lambda_k\{a_n\}_{n\in\mathbb{N}},\Lambda_k\{a_n\}_{n\in\mathbb{N}}\rangle_{\mathcal{A}} = \{0\}$$

So, we consider the operator K defined by,

$$K:\mathcal{H}\longrightarrow\mathcal{H}$$
$$\{a_n\}_{n\in\mathbb{N}}\longrightarrow\{a_{2k+1}\}_{k\in\mathbb{N}}$$

and $A=\sqrt{\alpha\beta}\left\{\frac{1}{\sqrt{k}}\right\}_{k\in\mathbb{N}^*}$, then,

$$A\langle K^*\{a_n\}_{n\in\mathbb{N}},K^*\{a_n\}_{n\in\mathbb{N}}\rangle_{\mathcal{A}}A^* = \sqrt{\alpha\beta}\left\{\frac{1}{\sqrt{k}}\right\}_{k\in\mathbb{N}^*}\left\{\frac{|a_{2k+1}|^2}{2k+1}\right\}_{k\in\mathbb{N}}\sqrt{\alpha\beta}\left\{\frac{1}{\sqrt{k}}\right\}_{k\in\mathbb{N}^*}$$
$$= \alpha\beta\left\{\frac{|a_{2k+1}|^2}{(2k+1)^2}\right\}_{k\in\mathbb{N}}$$
$$= \sum_{k\in\mathbb{N}}\langle\Lambda_k C\{a_n\}_{n\in\mathbb{N}},\Lambda_k C^{'}\{a_n\}_{n\in\mathbb{N}}\rangle_{\mathbb{A}}$$

Wich proof that $\{\Lambda_k\}_{k\in\mathbb{N}}$ is a tight $(C,C^{'})$-$*$-controlled operator frame.

Proposition 2.3. *Every $(C,C^{'})$-controlled $*$-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ is a $(C,C^{'})$-controlled $*$-K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.*

Proof. For any $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$, we have,

$$\langle K^{*}x, K^{*}x\rangle_{\mathcal{A}} \leq \|K\|^{2}\langle x, x\rangle_{\mathcal{A}}.$$

Let $\{T_i\}_{i\in I}$ be a $(C, C^{'})$-controlled $*$-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bounds A and B. Then,

$$A\langle x, x\rangle_{\mathcal{A}}A^{*} \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}B^{*}, x \in \mathcal{H}.$$

Hence,

$$A\|K\|^{-2}\langle K^{*}x, K^{*}x\rangle_{\mathcal{A}}A^{*} \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}B^{*}, x \in \mathcal{H}.$$

Thus

$$A\|K\|^{-1}\langle K^{*}x, K^{*}x\rangle_{\mathcal{A}}(A\|K\|^{-1})^{*} \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}B^{*}, x \in \mathcal{H}.$$

Therefore, $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled $*$ - K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bounds $A\|K\|^{-1}$ and B. □

Proposition 2.4. *Let $\{T_i\}_{i\in I}$ be a $(C, C^{'})$-controlled $*$-K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. If K is surjective then $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled $*$-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$.*

Proof. Suppose that K is surjective, from lemma 1.2 there exists $m > 0$ such that

$$\langle K^{*}x, K^{*}x\rangle_{\mathcal{A}} \geq m\langle x, x\rangle_{\mathcal{A}}, x \in \mathcal{H} \tag{2.2}$$

Let $\{T_i\}_{i\in I}$ be a $(C, C^{'})$-controlled $*$-K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bounds A and B, then,

$$A\langle K^{*}x, K^{*}x\rangle_{\mathcal{A}}A^{*} \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}B^{*}, x \in \mathcal{H}. \tag{2.3}$$

Using (2.2) and (2.3), we have

$$Am\langle x, x\rangle_{\mathcal{A}}A^{*} \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}B^{*}, x \in \mathcal{H}.$$

Then

$$A\sqrt{m}\langle x, x\rangle_{\mathcal{A}}(A\sqrt{m})^{*} \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}B^{*}, x \in \mathcal{H}.$$

Therefore $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled $*$-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. □

Proposition 2.5. *Let $C, C^{'} \in GL^{+}(\mathcal{H})$ and $\{T_i\}_{i\in I}$ be a $*$- K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. Assume that C and C' commute with T_i for each $i \in I$ and commute with K. Then $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled $*$-K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$.*

Proof. Let $\{T_i\}_{i\in I}$ be a $*$- K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$.
Then there exist two strictly non zero elements A and B in $\mathcal{A}$ such that

$$A\langle K^{*}x, K^{*}x\rangle_{\mathcal{A}}A^{*} \leq \sum_{i\in I}\langle T_ix, T_ix\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}B^{*}, x \in \mathcal{H}. \tag{2.4}$$

On one hand, we have,

$$\begin{aligned}\sum_{i\in I}\langle T_iCx,T_iC'x\rangle_{\mathcal{A}}&=\sum_{i\in I}\langle T_i(CC')^{\frac{1}{2}}x,T_i(CC')^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\&\leq B\langle (CC')^{\frac{1}{2}}x,(CC')^{\frac{1}{2}}x\rangle_{\mathcal{A}}\\&\leq B\|(CC')^{\frac{1}{2}}\|^2\langle x,x\rangle_{\mathcal{A}},\end{aligned}$$

So,

$$\sum_{i\in I}\langle T_iCx,T_iC'x\rangle_{\mathcal{A}}\leq B\|(CC')^{\frac{1}{2}}\|^2\langle x,x\rangle_{\mathcal{A}}. \tag{2.5}$$

Also, we have,

$$\begin{aligned}\sum_{i\in I}\langle T_i(CC')^{\frac{1}{2}}x,T_i(CC')^{\frac{1}{2}}x\rangle_{\mathcal{A}}&\geq A\langle K^*(CC')^{\frac{1}{2}}x,K^*(CC')^{\frac{1}{2}}x\rangle_{\mathcal{A}}A^*\\&\geq A\langle (CC')^{\frac{1}{2}}K^*x,(CC')^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}}A^*.\end{aligned}$$

Since $(CC')^{\frac{1}{2}}$ is a surjecive operator, then there exists $m>0$ such that,

$$\langle (CC')^{\frac{1}{2}}K^*x,(CC')^{\frac{1}{2}}K^*x\rangle_{\mathcal{A}}\geq m\langle K^*x,K^*x\rangle_{\mathcal{A}}. \tag{2.6}$$

Which give,

$$Am\langle K^*x,K^*x\rangle_{\mathcal{A}}A^*\leq\sum_{i\in I}\langle T_iCx,T_iC'x\rangle_{\mathcal{A}}\leq B\|(CC')^{\frac{1}{2}}\|^2\langle x,x\rangle_{\mathcal{A}}B^*.$$

Hence

$$A\sqrt{m}\langle K^*x,K^*x\rangle_{\mathcal{A}}(A\sqrt{m})^*\leq\sum_{i\in I}\langle T_iCx,T_iC'x\rangle_{\mathcal{A}}\leq B\|(CC')^{\frac{1}{2}}\|\langle x,x\rangle_{\mathcal{A}}(B\|(CC')^{\frac{1}{2}}\|)^*.$$

Therefore $\{T_i\}_{i\in I}$ is a $(C,C^{'})$-controlled $*$-K-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

□

Let $\{T_i\}_{i\in I}$ be a $(C,C')-$controlled Bessel $*$-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. We assume that C and C' commute between them and commute with $T_i^*T_i$ for each $i\in I$.
We define the operator $T_{(C,C')}:\mathcal{H}\to l^2(\mathcal{H})$ given by

$$T_{(C,C')}x=\{T_i(CC')^{\frac{1}{2}}x\}_{i\in I}.$$

$T_{(C,C')}$ is called the analysis operator. The adjoint operator for T is defined by $T_{(C,C')}^*:l^2(\mathcal{H})\to\mathcal{H}$ given by,

$$T_{(C,C')}^*(\{a_i\}_{i\in I})=\sum_{i\in I}(CC')^{\frac{1}{2}}T_i^*a_i$$

is called the synthesis operator.
We define the frame operator of the $(C,C^{'})$-controlled Bessel $*$-operator frame by:

$$\begin{aligned}S_{(C,C')}:&\mathcal{H}\longrightarrow\mathcal{H}\\&x\longrightarrow S_{(C,C')}x=T_{(C,C')}T_{(C,C')}^*x=\sum_{i\in I}C'T_i^*T_iCx.\end{aligned}$$

It's clear to see that $S_{(C,C')}$ is positive, bounded and selfadjoint.

Theorem 2.6. *Let $K, Q \in End_{\mathcal{A}}^{*}(\mathcal{H})$ and $\{T_i\}_{i\in I} \subset End_{\mathcal{A}}^{*}(\mathcal{H})$ be a $(C, C')-$controlled $*$-K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with frame operator $S_{(C,C')}$. Suppose that Q commute with C , C' and commute with K. Then $\{T_iQ\}_{i\in I}$ is a $(C, C')-$controlled $*$- (Q^*K)-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with frame operator $S = Q^*S_{(C,C')}Q$.*

Proof. Suppose that $\{T_i\}_{i\in I}$ is a $(C, C')-$controlled $*$- K-operator frame with frame bounds A and B. Then,

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}B^*.$$

Hence,

$$A\langle K^*Qx, K^*Qx\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle T_iCQx, T_iC^{'}Qx\rangle_{\mathcal{A}} \leq B\langle Qx, Qx\rangle_{\mathcal{A}}B^*.$$

So,

$$A\langle (Q^*K)^*x, (Q^*K)^*x\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle T_iQCx, T_iQC^{'}x\rangle_{\mathcal{A}} \leq B\|Q\|^2\langle x, x\rangle_{\mathcal{A}}B^*.$$

Then

$$A\langle (Q^*K)^*x, (Q^*K)^*x\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle T_iQCx, T_iQC^{'}x\rangle_{\mathcal{A}} \leq B\|Q\|\langle x, x\rangle_{\mathcal{A}}(B\|Q\|)^*.$$

Therefore $\{T_iQ\}_{i\in I}$ is a (C, C')-controlled $*$-(Q^*K)-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with bounds A and $B\|Q\|$.
Furthermore,

$$\begin{aligned} S = \sum_{i\in I} C'(T_iQ)^*T_iQCx &= \sum_{i\in I} C'Q^*T_i^*T_iCQx = Q^*\sum_{i\in I} C'T_i^*T_iCQx \\ &= Q^*S_{(C,C')}Qx \end{aligned}$$

□

Theorem 2.7. *Let $\{T_i\}_{i\in I}$ be a $(C, C')-$controlled $*$-K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with best frame bounds A and B. If $Q : \mathcal{H} \to \mathcal{H}$ is an adjointable and invertible operator such that Q and Q^{-1} commutes with C and $C^{'}$ for each $i \in I$ and Q^{-1} commute with K^*, then $\{T_iQ\}_{i\in I}$ is a $(C, C')-$controlled $*$-K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with best frame bounds M and N satisfying the inequalities,*

$$A\|Q^{-1}\|^{-1} \leq M \leq A\|Q\| \qquad \textit{and} \qquad A\|Q^{-1}\|^{-1} \leq N \leq B\|Q\|. \tag{2.7}$$

Proof. Let $\{T_i\}_{i\in I}$ be a $(C, C')-$controlled $*$-K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with best frame bounds A and B.

One one hand, we have

$$\begin{aligned} \sum_{i\in I}\langle T_iCQx, T_iC^{'}Qx\rangle_{\mathcal{A}} &\leq B\langle Qx, Qx\rangle_{\mathcal{A}}B^* \\ &\leq B\|Q\|^2\langle x, x\rangle_{\mathcal{A}}B^* \\ &\leq B\|Q\|\langle x, x\rangle_{\mathcal{A}}(B\|Q\|)^*. \end{aligned}$$

One the other hand, we have,

$$\begin{aligned} A\langle K^*x, K^*x\rangle_{\mathcal{A}}A^* &= A\langle K^*Q^{-1}Qx, K^*Q^{-1}Qx\rangle_{\mathcal{A}}A^* \\ &= A\langle Q^{-1}K^*Qx, Q^{-1}K^*Qx\rangle_{\mathcal{A}}A^* \\ &\leq A\|Q^{-1}\|^2\langle K^*Qx, K^*Qx\rangle_{\mathcal{A}}A^* \\ &\leq \|Q^{-1}\|^2\sum_{i\in I}\langle T_iCQx, T_iC^{'}Qx\rangle_{\mathcal{A}} \\ &= \|Q^{-1}\|^2\sum_{i\in I}\langle T_iQCx, T_iQC^{'}x\rangle_{\mathcal{A}}. \end{aligned}$$

Hence,

$$A\|Q^{-1}\|^{-2}\langle K^*x, K^*x\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle T_iQCx, T_iQC^{'}x\rangle_{\mathcal{A}} \leq B\|Q\|\langle x,x\rangle_{\mathcal{A}}(B\|Q\|)^*.$$

Thus

$$A\|Q^{-1}\|^{-1}\langle K^*x, K^*x\rangle_{\mathcal{A}}(A\|Q^{-1}\|^{-1})^* \leq \sum_{i\in I}\langle T_iQCx, T_iQC^{'}x\rangle_{\mathcal{A}} \leq B\|Q\|\langle x,x\rangle_{\mathcal{A}}(B\|Q\|)^*.$$

Therefore, $\{T_iQ\}_{i\in I}$ is a $(C,C')-$controlled $*$-K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with bounds $A\|Q^{-1}\|^{-1}$ and $B\|Q\|$.
Now, let M and N be the best bounds of the $(C,C')-$controlled $*$- K-operator frame $\{T_iQ\}_{i\in I}$. Then,

$$A\|Q^{-1}\|^{-1} \leq M \qquad and \qquad N \leq B\|Q\|. \tag{2.8}$$

Also, $\{T_iQ\}_{i\in I}$ is a $(C,C')-$controlled $*$-K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with frame bounds M and N.

Since,

$$\langle K^*x, K^*x\rangle_{\mathcal{A}} = \langle QQ^{-1}K^*x, QQ^{-1}K^*x\rangle_{\mathcal{A}} \leq \|Q\|^2\langle K^*Q^{-1}x, K^*Q^{-1}x\rangle_{\mathcal{A}}, x\in\mathcal{H}.$$

So,

$$\begin{aligned} M\|Q\|^{-2}\langle K^*x, K^*x\rangle_{\mathcal{A}}M^* &\leq M\langle K^*Q^{-1}x, K^*Q^{-1}x\rangle_{\mathcal{A}}M^* \\ &\leq \sum_{i\in I}\langle T_iQCQ^{-1}x, T_iQC'Q^{-1}x\rangle_{\mathcal{A}} \\ &= \sum_{i\in I}\langle T_iQQ^{-1}Cx, T_iQQ^{-1}C'x\rangle_{\mathcal{A}} \\ &= \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \\ &\leq N\|Q^{-1}\|^2\langle x,x\rangle_{\mathcal{A}} \\ &\leq N\|Q^{-1}\|\langle x,x\rangle_{\mathcal{A}}(N\|Q^{-1}\|)^*. \end{aligned}$$

Since A and B are the best bounds of $(C,C')-$controlled K-operator frame $\{T_i\}_{i\in I}$, we have

$$M\|Q\|^{-1} \leq A \qquad and \qquad B \leq N\|Q^{-1}\|. \tag{2.9}$$

Therfore the inequality (2.7) follows from (2.9) and (2.8). □

Theorem 2.8. *Let $(\mathcal{H}, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ and $(\mathcal{H}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ be two Hilbert C^*-modules and let φ $\mathcal{A} \longrightarrow \mathcal{B}$ be a $*$-homomorphisme and θ be a map on $\mathcal{H}$ such that $\langle \theta x, \theta y\rangle_{\mathcal{B}} = \varphi(\langle x, y\rangle_{\mathcal{A}}$ for all $x, y \in \mathcal{H}$. Suppose $\{T_i\}_{i\in I} \subset End^*_{\mathcal{A}}(\mathcal{H})$ is a $(C, C^{'})$-controlled $*$-K-operator frame for $(\mathcal{H}, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ with frame operator $S_{\mathcal{A}}$ and lower and upper bounds A and B respectively. If θ is surjective such that $\theta T_i = T_i\theta$ for each $i \in I$, $\theta C = C\theta$, $\theta C^{'} = C^{'}\theta$ and $\theta K^* = K^*\theta$, then $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled $*$-K-operator frame for $(\mathcal{H}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ with frame operator $S_{\mathcal{B}}$ and lower and upper bounds $\varphi(A)$, $\varphi(B)$ respectively and $\langle S_{\mathcal{B}}\theta x, \theta y\rangle_{\mathcal{B}} = \varphi(\langle S_{\mathcal{A}}x, y\rangle_{\mathcal{A}}$.*

Proof. Since θ is surjective, then for every $y \in \mathcal{H}$ there exists $x \in \mathcal{H}$ such that $\theta x = y$. Using the definition of $(C, C^{'})$-controlled $*$- K-operator frame we have,

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle \leq B\langle x, x\rangle_{\mathcal{A}}B^*, x \in \mathcal{H}.$$

We have for all $x \in \mathcal{H}$,

$$\varphi(A\langle K^*x, K^*x\rangle_{\mathcal{A}}A^*) \leq \varphi(\sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}}) \leq \varphi(B\langle x, x\rangle_{\mathcal{A}}B^*).$$

From the definition of $*$-homomorphisme we have

$$\varphi(A)\varphi(\langle K^*x, K^*x\rangle_{\mathcal{A}})\varphi(A^*) \leq \varphi(\sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}}) \leq \varphi(B)\varphi(\langle x, x\rangle_{\mathcal{A}})\varphi(B^*).$$

Using the relation betwen θ and φ we get

$$\varphi(A)\langle \theta K^*x, \theta K^*x\rangle_{\mathcal{B}}(\varphi(A))^* \leq \sum_{i\in I}\langle \theta T_iCx, \theta T_iC^{'}x\rangle_{\mathcal{B}} \leq \varphi(B)\langle \theta x, \theta x\rangle_{\mathcal{B}})(\varphi(B))^*.$$

Since $\theta T_i = T_i\theta$, $\theta C = C\theta, \theta C^{'} = C^{'}\theta$ and $\theta K^* = K^*\theta$ we have

$$\varphi(A)\langle K^*\theta x, K^*\theta x\rangle_{\mathcal{B}}(\varphi(A))^* \leq \sum_{i\in I}\langle T_iC\theta x, T_iC^{'}\theta x\rangle_{\mathcal{B}} \leq \varphi(B)\langle \theta x, \theta x\rangle_{\mathcal{B}})(\varphi(B))^*.$$

Therefore

$$\varphi(A)\langle K^*y, K^*y\rangle_{\mathcal{B}}(\varphi(A))^* \leq \sum_{i\in I}\langle T_iCy, T_iC^{'}y\rangle_{\mathcal{B}} \leq \varphi(B)\langle y, y\rangle_{\mathcal{B}})(\varphi(B))^*, y \in \mathcal{H}.$$

This implies that $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled $*$-K-operator frame for $(\mathcal{H}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ with bounds $\varphi(A)$ and $\varphi(B)$. Moreover we have

$$\begin{aligned}
\varphi(\langle S_{\mathcal{A}}x, y\rangle_{\mathcal{A}} &= \varphi(\langle \sum_{i\in I} T_iCx, T_iC^{'}y\rangle_{\mathcal{A}}) \\
&= \sum_{i\in I}\varphi(\langle T_iCx, T_iC^{'}y\rangle_{\mathcal{A}}) \\
&= \sum_{i\in I}\langle \theta T_iCx, \theta T_iC^{'}y\rangle_{\mathcal{B}} \\
&= \sum_{i\in I}\langle T_iC\theta x, T_iC^{'}\theta y\rangle_{\mathcal{B}} \\
&= \langle \sum_{i\in I} C^{'}T_i^*T_iC\theta x, \theta y\rangle_{\mathcal{B}} \\
&= \langle S_{\mathcal{B}}\theta x, \theta y\rangle_{\mathcal{A}}).
\end{aligned}$$

Which completes the proof. □

3. Operators preserving controlled $*$-K-operator frame

Proposition 3.1. *Let $K, U \in End_{\mathcal{A}}^{*}(\mathcal{H})$ such that $R(U) \subset R(K)$. If $\{T_i\}_{i\in I}$ is a $(C, C')-$controlled $*$-K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, then $\{T_i\}_{i\in I}$ is a $(C, C')-$controlled $*$-U-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$*

Proof. Assume that $\{T_i\}_{i\in I}$ is a $(C, C')-$controlled $*$-K-operator with bounds A and B, then

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}} A^* \leq \sum_{i\in I} \langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}} B^*, x \in \mathcal{H}. \tag{3.1}$$

Since $R(U) \subset R(K)$ then from lemma (1.4), there exists $\lambda > 0$ such that $UU^* \leq \lambda KK^*$. Using (3.1), we have

$$\frac{A}{\sqrt{\lambda}}\langle U^*x, U^*x\rangle_{\mathcal{A}}(\frac{A}{\sqrt{\lambda}})^* \leq \sum_{i\in I} \langle T_iCx, T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}} B^*, x \in \mathcal{H}.$$

Therefore $\{T_i\}_{i\in I}$ is a $(C, C')-$controlled $*$-U-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. □

Theorem 3.2. *Let $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$ with a dense range. Let $\{T_i\}_{i\in I}$ be a $(C, C')-$controlled $*$-K-operator and $U \in End_{\mathcal{A}}^{*}(\mathcal{H})$ have closed range and commute with C and $C^{'}$. If $\{T_iU\}_{i\in I}$ and $\{T_iU^*\}_{i\in I}$ are $(C, C')-$controlled $*$-K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ then U is invertible.*

Proof. Suppose that $\{T_iU\}_{i\in I}$ is a $(C, C')-$controlled $*$-K-operator frame with bounds A_1 and B_1, then

$$A_1\langle K^*x, K^*x\rangle_{\mathcal{A}} A_1^* \leq \sum_{i\in I} \langle T_iUCx, T_iUC^{'}x\rangle_{\mathcal{A}} \leq B_1\langle x, x\rangle_{\mathcal{A}} B_1^*, x \in \mathcal{H}. \tag{3.2}$$

Since K have a dense range then K^* is injective.
Hence from (3.2), $N(U) \subset N(K^*)$. Then U is injective.
Moreover $R(U^*) = N(U)^{\perp} = \mathcal{H}$. Therefore U is surjective.
Now assume that $\{T_iU^*\}_{i\in I}$ is a $(C, C')-$controlled $*$-K-operator frame with bounds A_2 and B_2, then

$$A_2\langle K^*x, K^*x\rangle_{\mathcal{A}} A_2^* \leq \sum_{i\in I} \langle T_iU^*Cx, T_iU^*C^{'}x\rangle_{\mathcal{A}} \leq B_2\langle x, x\rangle_{\mathcal{A}} B_2^*, x \in \mathcal{H}. \tag{3.3}$$

Hence U is injective, since $N(U^*) \subset N(K^*)$. Thus U is invertible. □

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