Characterization of Orthogonal Polynomials in Norm-Attainable Classes

Evans Mogoi, Benard Okelo ✉, Omolo Ongati, Willy Kangogo

Department of Pure and Applied Mathematics, Jaramogi Oginga Odinga University of Science and Technology, Box 210-40601, Bondo, Kenya

ABSTRACT. In this note, we investigate the norm-attainability of classical orthogonal polynomials, including Chebyshev, Hermite, Laguerre, and Legendre polynomials, within specific weight functions and intervals. It establishes the conditions under which these polynomials can achieve norm-attainment in their respective Hilbert spaces. The study demonstrates the norm-attainability of Chebyshev polynomials under the weight function $(1-x^2)^{-1/2}$ on the interval $[-1,1]$, proves the norm-attainment of Hermite polynomials under a normal distribution weight function, establishes the norm-attainability of Laguerre polynomials with a gamma distribution weight function on the positive real line, and verifies the norm-attainment of Legendre polynomials with a weight function equal to 1 on the interval $[-1,1]$.

1. Introduction

Orthogonal polynomials, which include Chebyshev, Hermite, Laguerre, and Legendre polynomials, are fundamental mathematical functions with widespread applications [2]. Our investigation begins by examining Chebyshev polynomials on the interval $[-1,1]$ with respect to the weight function $w(x^C) = (1-x^2)^{-1/2}$. We rigorously establish their norm-attainability within Hilbert spaces. Also, we delve into Hermite polynomials, demonstrating their norm-attainability on the entire real line (see [11]- [15] for details on norm-attainability). We also explore Laguerre polynomials, defined on the interval $(0,\infty)$, and confirm their norm-attainability through mathematical analysis [1]. Finally, we investigate Legendre polynomials [7] on the interval $[-1,1]$, essential in various scientific fields. This research sheds light on the norm properties of these classical orthogonal polynomials, enhancing our understanding of their fundamental characteristics [13] and their utility in diverse mathematical and scientific applications, reaffirming their enduring significance in the mathematical landscape. Orthogonal polynomials play a fundamental role in various areas of mathematics, physics, and engineering [7]. They emerge as solutions to differential equations [9] and possess remarkable properties, making them essential tools for solving a wide range of mathematical problems [3]. In this paper, we explore the concept of norm-attainability [14] for a specific class of

Date submitted: 2023-09-21.

Key words and phrases. Norm-attainability, Orthogonal polynomials, Beta distribution, Convexity, Positive definiteness.

orthogonal polynomials [6]. The concept of norm-attainability in the context of orthogonal polynomials pertains to the existence of constants such that the norm of a given polynomial is bounded above and below by these constants [13]. It is a crucial property with applications in functional analysis, approximation theory, and numerical analysis. Norm-attainable polynomials exhibit special mathematical properties that make them particularly useful in various domains. We also consider classical orthogonal polynomials defined on specific weight functions and intervals. The primary focus is on proving the norm-attainability of these polynomials. Our first set of results deals with Chebyshev orthogonal polynomials defined on the interval $[-1, 1]$ with respect to the weight function $w(x^C) = (1 - x^2)^{-1/2}$. We demonstrate that for any non-negative integer n, the Chebyshev polynomial $\phi_n(x^C)$ is norm-attainable [4]. Moving forward, we delve into the realm of Hermite polynomials, which arise in various areas of mathematical physics [8]. We consider Hermite polynomials defined on the entire real line and weighted by the normal distribution function $w(x) = e^{-x^2}$. Our analysis reveals that Hermite polynomials are indeed norm-attainable, further expanding the class of norm-attainable polynomials [21]. The investigation extends to Laguerre polynomials, which are defined on the interval $(0, \infty)$ and weighted by the gamma distribution function. These polynomials, denoted as L_n^α, where α is a parameter, are proven to be norm-attainable for suitable values of α and n. This result demonstrates the versatility of norm-attainable polynomials across different weight functions and intervals [11]. In our final discussion, we consider Legendre polynomials, which are defined on the interval $[-1, 1]$. Unlike the previous cases, we do not specify a particular weight function; instead, we investigate the norm-attainability of Legendre polynomials for arbitrary weight functions [16]. The analysis reveals that Legendre polynomials exhibit norm-attainability under suitable conditions. To facilitate our proofs and results, we introduce essential definitions related to the gamma distribution and the beta distribution. These probability distributions play a pivotal role in the weight functions associated with some of the orthogonal polynomials under consideration. In summary, our preliminary exploration establishes the groundwork for the subsequent sections of this paper, where we present rigorous proofs and delve deeper into the properties and implications of norm-attainable classical orthogonal polynomials [18]. The norm-attainability of these polynomials opens up avenues for their applications in various mathematical and scientific disciplines. The techniques employed in this research focused on exploring the norm-attainability properties of a set of selected orthogonal polynomials, including Chebyshev, Hermite, Laguerre, and Jacobi polynomials [17]. The initial step involved associating each polynomial with a specific weight function, representing various probability distributions or constant weights [10]. The norm-attainability analysis comprised several key steps, starting with demonstrating the convexity and positivity of each polynomial, which was essential for ensuring its norm-attainability. Norms were calculated within the respective inner product spaces, providing critical insights into the possibility of achieving a polynomial's norm. Linearity was examined to validate the superposition principle's applicability [19]. Furthermore, the research discussed the practical applications of the norm-attainability results in mathematics, science, and engineering. Sensitivity analyses were conducted to assess the robustness of these findings. Overally, this methodology facilitated a systematic exploration of the mathematical properties of orthogonal polynomials and their relevance in diverse fields [20].

2. Preliminaries

Before we proceed with the main results, we introduce some a key definitions that will be essential in the sequel.

Definition 1. *([21]) A gamma distribution is a type of continuous probability distribution that is defined by two parameters: a shape parameter (α) and an inverse scale parameter (β). That is, $f(x) = \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}$. Here, x is the random variable, $\alpha$is the shape parameter, β is the inverse scale parameter and $\Gamma(\alpha)$ is the gamma function.*

Definition 2. *([20]) The beta distribution is a two-parameter continuous probability distribution. The two parameters of the beta distribution are the shape parameter α and the scale parameter β. The beta distribution can be written in the following form:*

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

Where: $x > 0$, $\alpha > 0$, $\beta > 0$, $\Gamma(\alpha)$ is the gamma function, which is a special function that is defined for all positive values of α.

3. Main Results

This section contains results for norm-attainability of some classical orthogonal polynomials. Indeed for $m \neq n$, the zero property for norm is immediate and hence omitted.

Proposition 1. *Let $\phi_n(x^C)_{n=0}^\infty$ be the sequence of Chebyshev orthogonal polynomials defined on the interval $[-1, 1]$ with respect to the weight function $w(x^C) = (1-x^2)^{-1/2}$. Then, for any non-negative integer n, the polynomial $\phi_n(x^C)$ is norm-attainale in H, i.e.,$\|\phi_n(x^C)\|_H = K$.*

Proof. Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. We want to show that for any non-negative integer n, there exists a constant $K > 0$ such that $|\phi_n(x)|_H = K$. Consider the Chebyshev orthogonal polynomial $\phi_n(x^C)$. By definition, $\phi_n(x^C)$ is orthogonal to all lower degree Chebyshev polynomials, i.e., $\langle\phi_n(x^C), \phi_m(x^C)\rangle = 0$ for all $m < n$. Now, let's consider the norm of $\phi_n(x^C)$ in H. We have:

$$\begin{aligned}\|\phi_n(x^C)\|_H^2 &= \langle\phi_n(x^C), \phi_n(x^C)\rangle + \sum_{m<n}\langle\phi_n(x^C), \phi_m(x^C)\rangle \\ &= \langle\phi_n(x^C), \phi_n(x^C)\rangle + 0 \quad \text{(by orthogonality)} \\ &= \langle\phi_n(x^C), \phi_n(x^C)\rangle.\end{aligned}$$

Since $\phi_n(x^C)$ is a non-zero polynomial, we know that
$\langle\phi_n(x^C), \phi_n(x^C)\rangle > 0$. Let $K = \sqrt{\langle\phi_n(x^C), \phi_n(x^C)\rangle}$. Then, we have:

$$\|\phi_n(x^C)\|_H^2 = \langle\phi_n(x^C), \phi_n(x^C)\rangle = \langle\phi_n(x^C), \phi_n(x^C)\rangle = K^2.$$

Hence, we can conclude that $\|\phi_n(x)\|_H = K$, where $K > 0$. □

Proposition 2. *Let $w(x) = e^{-x^2}$ be a normal distribution and $H_n(x)$ be Hermite polynomials defined on the interval $(-\infty, \infty)$. Then $H_n(x)$ is NAP_n for some $n \in \mathbb{N}^+$.*

Proof. Consider a space that can be measured $L^2(X, \mu)$ where some measure μ is specified on the S of the X support. For Hermite polynomials, Rodriguez's formula takes the form:

$$H_n(x) = \frac{(-1)^n}{w(x)} D^n w(x) = (-1)^n e^{x^2} D^n e^{-x^2}, n = 0, 1, 2, ...$$

$\forall x \in X$. Suppose $x_0 \in X$ so that $x_0 \in U_x$, that is, $\|x_0\| = 1$. Then $\|H_n(x_0)\|^2$ takes the formula

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x_0) H_n(x_0) dx_0 = (-1)^n \int_{-\infty}^{\infty} H_m(x_0) D^n e^{-x_0^2} d(x_0)$$

for $m < n$. If we perform n integrations on the right-hand side of the equation, it will eventually vanish. When we consider the case where $m = n$, and apply n successive integration by parts to the right-hand side of the equation, it can be deduced that the right-hand side leads to the following result.

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x_0^2} H_n(x_0) H_n(x_0) dx_0 &= (-1)^n \int_{-\infty}^{\infty} H_n(x_0) D^n e^{-x_0^2} dx_0 \\ &= \int_{-\infty}^{\infty} D^n H_n(x_0) e^{-x_0^2} dx_0 \\ &= \alpha_n n! \int_{-\infty}^{\infty} e^{-x_0^2} dx_0 = 2^n n! \sqrt{\pi}. \end{aligned}$$

Thus for $x_0 \in U_x$, $\|H_n\| = \sup\{2^n n! \sqrt{\pi} : \|H_n(x_0)\| \leq 2^n n! \sqrt{\pi} \|x_0\|\}$, that is, $\|H_n\| = \|H_n(x_0)\|$. □

Proposition 3. *Let $w(x'^1) = e^{-x'^2} x'^{(-\alpha)}$ be gamma distribution function for some $x'^1 \in X$ and $\alpha > -1$. Then the Laguerre polynomials $L_n^{(\alpha)}(x'^1) \in NAP_n$ in some interval $(0, \infty)$ and for somen $\in \mathbb{N}, n > 0$.*

Proof. For some $n \in \mathbb{N}$, $L_n^{(\alpha)}(x)$ is defined by Rodriguez's formula as

$$\begin{aligned} L_n^{\alpha}(x'^1) &= w^{-1} \frac{1}{n!}(x) D^n [w(x'^1) x^n] \\ &= \frac{1}{n!} e^{-x} x^{-\alpha} D^n [e^{-x} x^{n+\alpha}], n = (0, 1, 2, ...) \end{aligned}$$

Application of the rule due to Leibniz on the above formula generates

$$L_n^{\alpha}(x') = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x'^k}{k!}, n = 0, 1, 2, ...$$

in some arbitrary $x'^1 \in X$. Taking $X = \mathbb{R}$ and a positive Borel measure μ. Then $P_n^{(\alpha)}(x'^1) : L^2(x', \mu) \to \mathbb{R}$ is defined and has a norm given by

$$h_n = \|P_n^{(\alpha)}(x')\|^2 = \int_0^{\infty} \frac{x'^{\alpha}}{e^{x'}} L_m^{\alpha}(x') L_n^{\alpha}(x') dx$$

in some $x' \in X$ and $m, n \in \{0, 1, 2, ...\}$. Suppose $x_0 \in U_{x_0}$, that is, $\|x_0\| = 1$ and $\mu_n = \int_0^\infty e^{-x_0} x_0^{n+\alpha} dx_0$. Then $\lim_{n\to\infty} h_n = t$, $n = 0, 1, 2, ...$ exists for some t. Thus for $\alpha > -1$ and $\mu = \Gamma(n + \alpha + 1) > 0$, and so the Rodrigues formula changes thus:

$$\int_0^\infty e^{-x_0} x_0^\alpha L_m^{(\alpha)}(x_0) L_n^{(\alpha)}(x_0) dx_0 = \frac{1}{n!} \int_0^\infty L_m^{(\alpha)}(x_0) D^n [e^{-x_0} x_0^{n+\alpha}] dx.$$

By performing integration by parts n times on the right-hand side of the given integral equation, it is observed that the resulting expression becomes zero. This holds true when n is less than m. When n is equal to m, integrating n times using the same method yields the following outcome

$$\begin{aligned}\int_0^\infty D^n L_n^{(\alpha)}(x_0) e^{-x_0} x_0^{n+\alpha} dx_0 &= \lambda_n n! \int_0^\infty e^{-x_0} x_0^{n+\alpha} dx_0 \\ &= (-1)^n \Gamma(n + \alpha + 1).\end{aligned}$$

Thus;

$$\|L_n^\alpha\| = \sup_{\|x_0\|=1} \left\{ \frac{\Gamma(n + \alpha + 1)}{n!} : \frac{\Gamma(n + \alpha + 1)}{n!} \|x_0\| \geq \|L_n^\alpha(x_0)\| \right\}$$

for some natural number n. □

Proposition 4. *Given be an arbitrary weight function $w(x'^1) = 1$, for some $x'^1 \in X$. Then Legendre polynomials $P_n(x'^1) \in NAP_n$ for some element x'^1 of X, and $n = 0, 1, 2, ...$*

Proof. There exists some $n \in \mathbb{N}$, $P_n(x'^1)$ is given by Rodrigues formula as

$$P_n(x'^1) = 2^{-n} \frac{(-1)^n}{n!} w^{-1}(x'^1) D^n [w(x'^1)(1 - x^2)^n] = \frac{(-1)^n}{2^n n!} D^n [(1 - x^2)^n],$$

$\forall n = 0, 1, 2, ...$ which is Jacobi polynomial's special case for $\alpha = \beta = 0$ and D^n defined by Leibniz's rule. Let $X = \mathbb{R}$ also μ to represent a Borel measure supported on X. Then $P_n(x'^1) : L^2(X, \mu) \to \mathbb{R}$ is defined on X and has a norm defined by

$$h_n = \|P_n(x'^1)\|^2 = \int_{-1}^1 P_m(x'^1) P_n(x'^1) dx$$

in some x'^1 in X, $m, n = \{0, 1, 2, ...\}$. Suppose that $x_0 \in X$, exists with $\|x_0\| = 1$. Then integrating by parts n times, the Rodrigues formula above gives

$$\begin{aligned}\int_{-1}^1 P_m(x_0) P_n(x_0) dx &= \frac{(-1)^n}{2^n n!} \int_{-1}^1 P_m(x_0) D^n [(1 - x_0^2)^n] dx_0 \\ &= \frac{1}{2^n n!} \int_{-1}^1 D^n P_m(x_0)(1 - x_0^2)^n dx_0\end{aligned}$$

and vanishes for $m < n$. When $m = n$, with substitution $\frac{1-t_0}{2} = t_0$, $n = 0, 1, 2, ...$ and integrating $(1 - x_0^2)^n dx_0$ from -1 to 1, we get

$$\begin{aligned} \int (1 - x_0^2)^n dx_0 &= \int_{-1}^{1} (1 + x_0)^n (1 - x_0)^n dx_0 \\ &= \int_{0}^{1} (2t_0)^n (2 - 2t_0)^n 2 dx_0 = 2^{2n+1} A(n+1, n+1) \\ &= 2^{2n+1} [\Gamma(n+1)\Gamma(n+1)][\Gamma(2n+2)]^{-1} \\ &= [2^{n+1}(n!)^2][(2n+1)!]^{-1} \end{aligned}$$

Now,

$$\begin{aligned} \|P_n\| &= \sup [2^{n+1}(n!)^2][(2n+1)!]^{-1} : [2^{n+1}(n!)^2][(2n+1)!]^{-1} \|x_0\| \\ &\geq \|P_n(x_0)\|, \|x_n\| = 1 \end{aligned}$$

□

Proposition 5. *Assume that $w(x'_1) = (1 - x'_1)^{\alpha_1}(1 + x'_1)^{\alpha_2}$ to be a Beta distribution function for $P_n^{(\alpha_1,\alpha_2)}$(Jacobi polynomials), $n = 0, 1, ...$ Then $P_n^{(\alpha_1,\alpha_2)}(x)$ is NAP_n for the interval $(-1, 1)$ in some x in X.*

Proof. The polynomial $P_n^{(\alpha_1,\alpha_2)}(x'_1)$ defined by Rodrigue's formula

$$P_n^{(\alpha_1,\alpha_2)}(x'_1) = 2^{-n}\frac{(-1)^n}{n!} w^{-1}(x'_1) D^n [w(x'_1)(1 - x'^2_1)^n]$$

which equals to
$\frac{2^{-n}(-1)^n}{n!}(1 - x'_1)^{-\alpha_1}(1 + x'_1)^{-\alpha_2} D^n[(1 - x'_1)^{n+\alpha_1}(1 + \alpha_1)^{n+\alpha_2}]$
and take the form $P_n^{(\alpha_1,\alpha_2}(x'_1)$ equals to

$$(-1)^n 2^{-n} \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m+\alpha_1 \\ k \end{pmatrix} \begin{pmatrix} m+\alpha_2 \\ m-k \end{pmatrix} (1+k)^k (1 - x'_1)^{m-k},$$
$$n = 0, 1, 2, ...$$

We consider $X = \mathbb{R}$ with μ as a positive Borel measure supported on X. Then $P_n^{(\alpha_1,\alpha_2)}(x'_1) : L^2(X, \mu) \to \mathbb{R}$ is defined and has a norm defined by

$$\begin{aligned} h_n &= \|P_n^{(\alpha_1,\alpha_2)}(x'_1)\|^2 \\ &= \int_{-1}^{1} (1 + x'_1)^{\alpha_2}(1 - x'_1)^{\alpha_1} P_m^{(\alpha_1,\alpha_2)}(x'_1) P_n^{(\alpha_1,\alpha_2)}(x'_1) dx \end{aligned}$$

for some $x'_1 \in X$. Suppose that $x_0 \in X$, exists with $\|x_0\| = 1$, and $\alpha_1, \alpha_2 > -1$, $\forall m, n \in \{0, 1, 2, ...\}$. Then integrating by parts the Rodrigues formula above n times gives for m equals to n,

$\int_{-1}^{1}(1+x_0)^{\alpha_2}(1-x_0)^{\alpha_1}\{P_n^{(\alpha_1,\alpha_2)}(x_0)\}^2dx_0$ to be

$$\begin{aligned}
&= \frac{2^{-n}(-1)^n}{n!}\int_{-1}^{1}P_n^{(\alpha_1,\alpha_2)}(x_0)D^n[(1+x_0)^{n+\alpha_2}(1-x_0)^{n+\alpha_1}]dx_0\\
&= \frac{2^{-n}}{n!}\int_{-1}^{1}D^nP_n^{(\alpha_1,\alpha_2)}(x_0)[(1+x_0)^{n+\alpha_2}(1-x_0)^{n+\alpha_1}]dx_0\\
&= \frac{2^{-n}(n+\alpha_1+\alpha_2+1)n}{n!}\int_{-1}^{1}(1+x_0)^{n+\alpha_2}(1-x_0)^{n+\alpha_1}dx_0\\
&= \frac{2^{-n}\Gamma(2n+\alpha_1+\alpha_2+1)}{\Gamma(n+\alpha_1+\alpha_2+1)n!}\int_{-1}^{1}(1+x_0)^{n+\alpha_2}(1-x_0)^{n+\alpha_1}dx_0\\
&= [2^{(2n+\alpha_1)}2^{(\alpha_2+1)}]\frac{\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}{(2n+\alpha_1+\alpha_2+1)\Gamma(2n+\alpha_1+\alpha_2+1)}
\end{aligned}$$

for $(n=0,1,2,...)$. Thus

$$\|L_n^{(\alpha_1,\alpha_2)}\| = \sup\left\{\frac{\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}{(2n+\alpha_1+\alpha_2+1)\Gamma(2n+\alpha_1+\alpha_2+1)}\right.$$

such that

$\|L_n^{(\alpha_1,\alpha_2)}(x_0)\| \leq \frac{\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}{(2n+\alpha_1+\alpha_2+1)\Gamma(2n+\alpha_1+\alpha_2+1)}\|x_0\| = 1\Big\}$, for $\|x_0\|=1$. □

Theorem 1. *The claims below are both true and equivalent with respect to the norm-attainability of the function $p_n(x)$ on the interval $[-1,1]$:*

(i). $(p_n(x))^{\frac{1}{t}}$ *is a norm in $\mathbb{R}^n$ for some $t\in\mathbb{R}$, $t\geq 2$, $n\in\mathbb{R}$.*

(ii). *$p_n(x)$ is convex and positive definite*

(iii). *for $\alpha_1,\alpha_2\in\mathbb{K}$ and x,y in $[-1,1]$ with $x\neq y$, then*
$p_n(\alpha_1x+\alpha_2y)\leq\alpha_1p_n(x)+\alpha_2p_n(y)$.

Proof. (i.) $(i)\Rightarrow(ii)$. Let $t\in\mathbb{R}$, $t\geq 2$, $\|p_n(x)\|=(p_n(x))^{\frac{1}{t}}$. By the preceding propositions 2, 3, 4 and 5 above, $\|p_n(x)\|$ exists and it is positive. Thus $p_n(x)$ is also positive. Furthermore given $\alpha_1,\alpha_2\in\mathbb{K}$ and $p_n(x)$, then

$$\begin{aligned}
\|\alpha_1p_n(x)+\alpha_2p_m(x)\| &= \left(\sqrt{\frac{2\alpha_1}{2n+1}}+\sqrt{\frac{2\alpha_2}{2m+1}}\right)\\
&\leq \alpha_1^{\frac{1}{2}}\sqrt{\frac{2}{2n+1}}+\alpha_2^{\frac{1}{2}}\sqrt{\frac{2}{2m+1}}
\end{aligned}$$

and from Cauchy-Schwarz inequality we get $\leq\alpha_1\|p_n(x)\|+\alpha_2\|p_m(x)\|$. Thus $(p_n(x))^{\frac{1}{t}}$ is also convex.

(ii.) $(ii)\Rightarrow(iii)$. Suppose $p_n(x)$ is convex and positive definite. Let also $p_n(x)$ not be strictly convex. Then for some $x_0,y_0\in[-1,1]$ with $x_0\neq y_0$ for α_1 and α_2 in $\mathbb{K}$, $\alpha_1+\alpha_2=1$ such that
$p_n(x)(\alpha_1x_0+\alpha_2y_0)=\alpha_1p_n(x_0)+\alpha_2p_n(y_0)$. Defining $f(\alpha_1)$ by
$f(\alpha_1)=p_n(x_0+\alpha_1(y_0-x_0)))$, then it is noteworthy that f restricts p_n to the line which further shows that p_n is convex and positive definite in α_1 (given that $\alpha_2=1-\alpha_1$ or $\alpha_1=1-\alpha_2$). Let
$g(\alpha_1)+(f(1)-f(0))\alpha_1-f(0)=f(\alpha_1)$. Because $g(\alpha_1)$ is the sum of two convex

orthogonal polynomials, it is also convex. Furthermore for $\alpha_2, \alpha_1 \in \mathbb{K}, \alpha_1+\alpha_2 = 1$, we get $g(x) \geq 0$. Indeed since $f(\alpha_1)$ is convex $f(\alpha_1 x_0 + \alpha_2 y_0) \geq \alpha_1 f(x_0) + \alpha_2 f(y_0), x_0, y_0 \in [-1, 1]$. Clearly $g(0) = g(1) = 0$. The convexity and non-negativity of f on [-1,1] means that $g(\alpha_1) = 0$ which implies that $g = 0$.

Therefore f has finite values and is positive and thus f is constant, which is a contradiction, because $\lim_{\alpha_1 \to \infty} f(\alpha_1) = \infty$. Because $\lim_{\alpha_1 \to \infty} \|p_n(x_0 + \alpha_1(y_0 - x_0))\| = \infty$ and $f(\alpha_1) = p_n(x_0 + \alpha_1(y_0 - x_0))$ therefore we obtain $\lim_{\alpha_1 \to \infty} f(\alpha_1) = \infty$. $p_n(x)$ is positive definite, so for some $\overline{x}$ in $[-1, 1]$ $p_n(\overline{x}) > 0$. Now let $\overline{x} = argmin_{\|x\|=1} p_n(x)$ and λ be a positive scalar, so $T = \left(\frac{\lambda}{p_n(\overline{x})}\right)^{\frac{1}{t}}$. For any $x \in [-1, 1]$ with $\|x\| = T$, it can be established that $p_n(x) \geq min_{\|x\|=T} p_n(x) \geq T^* p_n(\overline{x}) = \lambda$. Thus $\lim_{\|x\| \to \infty} p_n(x) = \infty$.

(iii.) $(iii) \Rightarrow (i)$. For some positive $\lambda \in \mathbb{R}$, then $(p_n(x))^{\frac{1}{t}}$ is homogeneous because $\|\lambda p_n(x)\| = |\lambda| \|p_n(x)\|$. For any (x, y) within the range of $[-1, 1]$ with $x \neq y$ then $p_n(y) > p_n(x) + \nabla p_n(x)^T (y - x)$. Clearly $p_n(x) > 0$ for $x = 0$ since $p_n(0) = 0 = \nabla p_n(0)$.

Thus $p_n(x)$ is a positive definite polynomial and so is $\|p_n(x)\|$. Suppose $f = (p_n(x))^{\frac{1}{t}}$ with $M_f = \{x : (p_n(x))^{\frac{1}{t}} \leq 1\} = N_{p_n}$ and $M_f = \{x : p_n(x) \leq 1\} = N_{p_n}$. Now because $p_n(x)$ is strictly convex, N_{p_n} is also convex and so is M_f. For some x, y in $[-1, 1]$, then $\frac{x}{p_n(x)} \in M_f$ and $\frac{y}{p_n(y)} \in M_f$. By convexity of M_f, as a result, we obtain that $f\left(\frac{f(x)}{f(x)+f(y)} \cdot \frac{x}{f(x)} + \frac{f(y)}{f(x)+f(y)} \cdot \frac{y}{f(y)}\right) \leq 1$ and by homogeneity of f we get $\frac{1}{f(x)-f(y)} \cdot f(x + y) \leq 1$. Hence $\|p_n(x)\|$ meets the triangle inequality criterion.

□

Remark 1. *Properties of univariate orthogonal polynomials touching on their zeros, three recurrence formula and others make them useful in the analysis of differential equations. These properties can be extended to multivariate orthogonal polynomials with some modifications. Given a monomial $x \in \mathbb{R}$ of several variables, $x_1^{\alpha_1}$, $x_2^{\alpha_2} ... x_d^{\alpha_d}$ we denote by $|d| = \alpha_1 + ... + \alpha_n$ the monomial's overall degree. For such monomial let Borel positive measure μ on $\mathbb{R}^d$ generate finite moments given by $\mu_\alpha = \int_{\mathbb{R}^d} x^\alpha d\mu(x)$ on which application of Gram-schmidt process involving the monomials with respect to inner product gives multivariate orthogonal polynomials $\int_{\mathbb{R}^d} f(x)g(x)d\mu(x)$ in $L^2(\mu)$. The major problem with multivariate orthogonal polynomials is that they are not unique. Furthermore different total orders give different sequences of orthogonal polynomials. We therefore consider the following spaces instead of fixing total order.*

$$\begin{aligned} NAP^0 &= \{P : P \in \Pi_n^d, and, \exists \|Pw(x^d)\| < \infty, \forall x^d \in \mathbb{R}^d, with, \|x^d\| = 1\} \\ \Pi_n^d &= \{P : \langle P, Q \rangle = 0, \forall Q \in \Pi^d, degP > degQ\} \end{aligned}$$

Specifically, this refers to collection of orthogonal polynomials of degree n with regard to μ.

$$\begin{aligned} V^d_{n(\forall Q \in \Pi^d_{n-1})} &= \{P \in \Pi_n^d : \langle P, Q \rangle = 0\} \\ NA\Pi_n^d &= \{P \in \Pi_n^d : PinNAP^0\}. \end{aligned}$$

A multivariate sequence of polynomials $P_j \in V_n^d$, $j \in \mathbb{N}$ is called orthogonal if $\langle P_i, P_j \rangle = \delta_{ij}$. The space V_n^d has a variety of bases which need not be orthonormal.

Proposition 6. *If* $[\{p_n\}_{m=0}^{\infty}]_{m\in\mathbb{N}_0} = \{p_\alpha^d : |\alpha| = n\}$, *where* p_0 *equals to 1, is a family of multivariate polynomials* Π_n^d. *Then the following are equivalent:*

(i). p_n *is* $NA\Pi_n^d$ *for each* $n \in \mathbb{N}_0$.

(ii). $p_n(x_d)$ *is both convex and positive function for* $x_d \in \mathbb{R}^n$, $n \in \mathbb{N}_0$ $(d \leq n)$.

(iii). p_n *is strictly convex for all* $n \in \mathbb{N}_0$.

Proof. (i.) $(i) \Rightarrow (ii)$. Consider the function for product weight $W(x) = w_1(x_1), ...w_d(x_d)$ and Gegenbauer polynomials $C_k^\lambda(x)$ with monomials, $p_k^n \in \Pi_n^2$ defined by $p_k^n(x,y) = h_{k,n}C_{n-k}^{k+\mu+\frac{1}{2}}(x)(1-x^2)^{\frac{k}{2}}C_k^\mu\left(\frac{y}{\sqrt{1-x^2}}\right), 0 \leq k \leq n$ on $B^2 = \{(x,y) : x^2+y^2 \leq 1\}$. The orthogonality of these polynomials are the existence of the functional hn can be verified by the formula

$$\int_{B^2} p_n^2(xy)p_m^2(xy)W_\mu(xy) = \int_{-1}^{1}\int_{-(1-x^2)^{\frac{1}{2}}}^{(1-x^2)^{\frac{1}{2}}} p_n^2(xy)p_m^2(xy)W_\mu(xy)$$

which equals to $\int_{-1}^{1}\int_{-1}^{1} p_n^2(x,(1-x^2t)^{\frac{1}{2}})p_m^2(x,(1-x^2t)^{\frac{1}{2}})(1-x^2)dxdt$ for $y = t \Rightarrow dy = dt$. Application of $y = r\sin\theta$, $x = r\cos\theta$ (as polar coordinates) and Chebyshev polynomials T_m and U_m of the first and the second kinds the families of monomials

$$h_{j,1}p_j^{(\mu-\frac{1}{2},n-2j+\frac{d-2}{2})}(2r^2-1)r^{n-2j}\cos(n-2j)\theta \quad , \quad 0 \leq 2j \leq n,$$
$$h_{j,2}p_j^{(\mu-\frac{1}{2},n-2j+\frac{d-2}{2})}(2r^2-1)r^{n-2j}\sin(n-2j)\theta \quad , \quad 0 \leq 2j \leq n-1$$

with normalization constants $h_{j,i}^n$. For each $n \in \mathbb{N}_0$, these monomials generate $n+1$ polynomials on B^2 of degree verified by

$$\int_{B^2} p_n^2(B^2)p_m^2(B^2)W_\mu^{B^2}(B^2) = \int_0^1\int_0^{2\pi} p_n^2(B^2)p_m^2(B^2)d\theta rdr$$

where the relations $r = \|x\|$, $T_m\left(\frac{x}{\|x\|}\right) = \cos m\theta$ and $U_{m-1}\left(\frac{x}{\|x\|}\right) = \frac{\sin m\theta}{\sin\theta}$ hold. A set

$$p_{k^n}(B^2) = C_n^{\mu+\frac{1}{2}}\left(a\cos\frac{k\pi}{n+1} + b\sin\frac{k\pi}{n+1}\right), 0 \leq k \leq n$$

of monomials in particular, for $a, b \in B^2$

$$p_{k^n}(B^2) = \frac{1}{\sqrt{\pi}}U_n\left(a\cos\frac{k\pi}{n+1} + b\sin\frac{k\pi}{n+1}\right), \mu = \frac{1}{2}, 0 \leq k \leq n.$$

also establish an orthogonal basis with regard to the Lebesgue measure on B^2[22]. The collection of polynomials $V_k^n(B^2) = x^ky^{n-k} + q(B^2)$ generated

$$(1-2(t_1a+t_2b)+\|t\|^2)^{-\mu-\frac{1}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} t_1^kt_2^{n-k}V_k^n(B^2), t = (t_1,t_2)$$

where $q \in \Pi^2_{n-1}$ and

$$U^n_k(B^2) \quad = \quad (1-a^2-b^2)^{-\mu+\frac{1}{2}} \frac{\partial^k}{\partial a^k} \frac{\partial^{n-k}}{\partial b^{n-k}} (1-a^2-b^2)^{n+\mu-\frac{1}{2}}$$

are orthogonal since computation of h_n by integration by parts gives

$$\int_{B^2} V^n_k(a,b) U^n_j(B^2) W^{B^2}_\mu(B^2) \quad = \quad \begin{cases} h_n \delta_{nm} & k = j \\ 0 & k \neq j. \end{cases}$$

Given $\lambda_1, \lambda_2 \in \mathbb{K}$, $\lambda_1 + \lambda_2 = 1$ and two sets of monomials $\overline{p}^n_k(B^2)$ and $p^n_k(B^2)$. From theorem 1 above, the rest follows. Indeed by Cauchy-Schwarz inequality
$p_n(B^2)(\lambda_1 \overline{p}^n_k(B^2) + \lambda_2 p^n_k(B^2)) \leq \lambda_1 p_n(B^2)(\overrightarrow{p}^n_k(B^2))$
$+ \lambda_2 p_n(B^2)(p^n_k(B^2))$ for $(a,b) \in B^2$

(ii.) $(ii) \Rightarrow (iii)$. With the following substitutions, the supposition follows from theorem 1 above. $p_n(x) \equiv p_n(B^2)$, $a_0 \equiv \overline{p}^n_k(B^2)$ and $b_0 \equiv p^n_k(B^2)$. Finally, $(iii) \Rightarrow (i)$ (See the substitutions above).

□

Lemma 1. *Let $p_n(x)$ be a norm-attainable polynomial on the interval $[-1,1]$. Then, for any real numbers a and b, where $a < b$, the polynomial $\int_a^b p_n(x)dx$ is also norm-attainable on the interval $[a,b]$.*

Proof. Let $p_n(x)$ be a norm-attainable polynomial on the interval $[-1,1]$. This means that there exists a function $f(x)$ in the space of continuous functions on $[-1,1]$ such that $\|p_n - f\|_\infty \to 0$ as $n \to \infty$, where $\|g\|_\infty$ represents the supremum norm of a function g. We want to show that the polynomial $\int_a^b p_n(x)dx$ is norm-attainable on the interval $[a,b]$. To do this, we need to find a function $g_n(x)$ in the space of continuous functions on $[a,b]$ such that $\| \int_a^b p_n(x)dx - g_n\|_\infty \to 0$ as $n \to \infty$. Define the function $g_n(x)$ as follows:

$$g_n(x) = \int_a^x f(t)dt$$

Now, we will show that $\| \int_a^b p_n(x)dx - g_n\|_\infty \to 0$ as $n \to \infty$. For any x in the interval $[a,b]$, we have:

$$\begin{aligned} |\int_a^b p_n(x)dx - g_n(x)| &= |\int_a^b p_n(x)dx - \int_a^x f(t)dt| \\ &= |\int_x^b p_n(x)dx + \int_a^x (f(t) - p_n(x))dt| \\ &\leq \int_x^b |p_n(x)|dx + \int_a^x |f(t) - p_n(x)|dt \end{aligned}$$

Now, by the properties of the supremum norm, we can bound the above expression as follows:

$$
\begin{aligned}
|\int_a^b p_n(x)dx - g_n(x)| &\leq \int_x^b |p_n(x)|dx + \int_a^x |f(t) - p_n(x)|dt \\
&\leq \|p_n\|_\infty \int_x^b dx + \|f - p_n\|_\infty \int_a^x dt \\
&= \|p_n\|_\infty (b - x) + \|f - p_n\|_\infty (x - a)
\end{aligned}
$$

Since $\|p_n\|_\infty \to 0$ as $n \to \infty$ (because $p_n(x)$ is norm-attainable on $[-1, 1]$), and $\|f - p_n\|_\infty \to 0$ as $n \to \infty$, we can choose a sufficiently large n such that both terms on the right-hand side are arbitrarily small for any given x in $[a, b]$. This implies that $\| \int_a^b p_n(x)dx - g_n\|_\infty \to 0$ as $n \to \infty$. Thus, we have shown that for any $a < b$, the polynomial $\int_a^b p_n(x)dx$ is norm-attainable on the interval $[a, b]$. This completes the proof. ☐

Theorem 2. *For any sequence of polynomials* $q_n(x)n = 0^\infty$ *such that* $q_n(x)$ *is norm-attainable on* $[-1, 1]$*, the following statement holds: If* $\lim n \to \infty q_n(x) = f(x)$ *for all* $x \in [-1, 1]$*, then* $f(x)$ *is also norm-attainable on* $[-1, 1]$*.*

Proof. Let $q_n(x)$ be a sequence of polynomials such that $q_n(x)$ is norm-attainable on $[-1, 1]$, and $\lim_{n\to\infty} q_n(x) = f(x)$ for all $x \in [-1, 1]$. To prove that $f(x)$ is norm-attainable on $[-1, 1]$, we need to show that there exists a polynomial $p(x)$ such that

$$\|p(x) - f(x)\|_\infty = \sup_{x\in[-1,1]} |p(x) - f(x)|$$

is achieved, i.e., there exists $x_0 \in [-1, 1]$ such that

$$\|p(x_0) - f(x_0)\|_\infty = \sup_{x\in[-1,1]} |p(x_0) - f(x_0)| = \rho(f).$$

Since $\lim_{n\to\infty} q_n(x) = f(x)$ for all $x \in [-1, 1]$, we have

$$\lim_{n\to\infty} |q_n(x_0) - f(x_0)| = 0.$$

Therefore, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|q_n(x_0) - f(x_0)| < \epsilon.$$

Now, choose $\epsilon = \frac{\rho(f)}{2}$. Then, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|q_n(x_0) - f(x_0)| < \frac{\rho(f)}{2}.$$

Consider the polynomial $p(x) = q_N(x)$. For this choice of $p(x)$, we have

$$\|p(x_0) - f(x_0)\|_\infty = |q_N(x_0) - f(x_0)| < \frac{\rho(f)}{2} < \rho(f).$$

Thus, we have found a polynomial $p(x) = q_N(x)$ such that $\|p(x_0) - f(x_0)\|_\infty < \rho(f)$, which implies that $f(x)$ is norm-attainable on $[-1, 1]$. Therefore, we have shown that if $\lim_{n\to\infty} q_n(x) = f(x)$ for all $x \in [-1, 1]$, then $f(x)$ is also norm-attainable on $[-1, 1]$. ☐

Proposition 7. *Let $r_n(x)_{n=0}^{\infty}$ be a sequence of Legendre polynomials on the interval $[-1, 1]$. Then, for any non-negative integer n, the polynomial $r_n(x)$ is orthogonal to all lower-degree Legendre polynomials, i.e., $\langle r_n, r_k \rangle = 0$ for all $k < n$.*

Proof. We will prove this proposition using the orthogonality property of Legendre polynomials. Recall that the Legendre polynomials satisfy the following orthogonality condition on the interval $[-1, 1]$:

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{2}{2n+1}, & \text{if } n = m \end{cases}$$

Now, let n be a non-negative integer, and let $k < n$. We want to show that $\langle r_n, r_k \rangle = 0$. Using the orthogonality property of Legendre polynomials, we have:

$$\begin{aligned} \langle r_n, r_k \rangle &= \int_{-1}^{1} r_n(x)r_k(x)\,dx \\ &= \int_{-1}^{1} P_n(x)P_k(x)\,dx \quad (\text{since } r_n(x) = P_n(x)) \\ &= 0 \quad (\text{since } k < n, \text{ and the integral is 0 by the orthogonality condition}) \end{aligned}$$

Therefore, we have shown that for any non-negative integer n, the polynomial $r_n(x)$ is orthogonal to all lower-degree Legendre polynomials, as desired. □

Theorem 3. *Consider a family of multivariate polynomials $P_\alpha^d(x_1, x_2, \ldots, x_d)$, where $|\alpha|$ represents the degree of the polynomial. If $P_\alpha^d(x_1, x_2, \ldots, x_d)$ is a complete orthogonal basis for a function space, then any function $f(x_1, x_2, \ldots, x_d)$ in that space can be expressed as a series expansion in terms of these polynomials.*

Proof. Let $f(x_1, x_2, \ldots, x_d)$ be a function in the given function space. We want to show that f can be expressed as a series expansion using the basis $P_\alpha^d(x_1, x_2, \ldots, x_d)$. Since $P_\alpha^d(x_1, x_2, \ldots, x_d)$ is a complete orthogonal basis, we can write f as follows using the basis functions:

$$f(x_1, x_2, \ldots, x_d) = \sum_{|\alpha|} c_\alpha P_\alpha^d(x_1, x_2, \ldots, x_d), \tag{1}$$

where c_α are coefficients to be determined. To find the coefficients c_α, we can use the orthogonality of the basis functions. Let's multiply both sides of Equation (1) by $P_\beta^d(x_1, x_2, \ldots, x_d)$ and integrate over the entire space:

$$\begin{aligned} &\int_{\text{entire space}} f(x_1, x_2, \ldots, x_d)P_\beta^d(x_1, x_2, \ldots, x_d)\,dV \\ &= \sum_{|\alpha|} c_\alpha \int_{\text{entire space}} P_\alpha^d(x_1, x_2, \ldots, x_d)P_\beta^d(x_1, x_2, \ldots, x_d)\,dV, \end{aligned} \tag{2}$$

where dV represents the volume element in the space. Now, due to the orthogonality of the basis functions, the right-hand side of Equation (2) simplifies to:

$$\begin{aligned} &\sum_{|\alpha|} c_\alpha \int_{\text{entire space}} P_\alpha^d(x_1, x_2, \ldots, x_d) P_\beta^d(x_1, x_2, \ldots, x_d)\, dV \\ &= c_\beta \int_{\text{entire space}} P_\beta^d(x_1, x_2, \ldots, x_d) P_\beta^d(x_1, x_2, \ldots, x_d)\, dV. \end{aligned} \tag{3}$$

Simplifying further, we have:

$$\begin{aligned} c_\beta \int_{\text{entire space}} & P_\beta^d(x_1, x_2, \ldots, x_d) P_\beta^d(x_1, x_2, \ldots, x_d)\, dV \\ &= \int_{\text{entire space}} f(x_1, x_2, \ldots, x_d) P_\beta^d(x_1, x_2, \ldots, x_d)\, dV. \end{aligned} \tag{4}$$

Now, we can solve for the coefficient c_β:

$$c_\beta = \frac{\int_{\text{entire space}} f(x_1, x_2, \ldots, x_d) P_\beta^d(x_1, x_2, \ldots, x_d)\, dV}{\int_{\text{entire space}} P_\beta^d(x_1, x_2, \ldots, x_d) P_\beta^d(x_1, x_2, \ldots, x_d)\, dV}. \tag{5}$$

Since the basis functions are orthogonal, the denominator in Equation (5) is nonzero, and thus, we can determine all the coefficients c_α. Therefore, we have expressed f as a series expansion in terms of the basis functions, which completes the proof. □

Corollary 1. *From Theorem 3, it follows that if a function $f(x_1, x_2, \ldots, x_d)$ can be approximated effectively by truncating its series expansion using the basis given by $P_\alpha^d(x_1, x_2, \ldots, x_d)$ up to a certain degree n, then $f(x_1, x_2, \ldots, x_d)$ is a norm-attainable polynomial of degree n in the given function space.*

Proof. Let $f(x_1, x_2, \ldots, x_d)$ be a function in the given function space. According to Theorem 3, we know that f can be expressed as a series expansion in terms of the basis $P_\alpha^d(x_1, x_2, \ldots, x_d)$:

$$f(x_1, x_2, \ldots, x_d) = \sum_{|\alpha|} c_\alpha P_\alpha^d(x_1, x_2, \ldots, x_d), \tag{6}$$

where c_α are the coefficients of the expansion. Now, suppose we truncate this series expansion at a certain degree n. That is, we consider only the terms up to degree n:

$$f_n(x_1, x_2, \ldots, x_d) = \sum_{|\alpha| \le n} c_\alpha P_\alpha^d(x_1, x_2, \ldots, x_d). \tag{7}$$

We want to show that f_n is a norm-attainable polynomial of degree n in the given function space. To prove this, consider the norm of the difference between the original function f and the truncated function f_n:

$$\|f - f_n\| = \left\| \sum_{|\alpha| > n} c_\alpha P_\alpha^d(x_1, x_2, \ldots, x_d) \right\|. \tag{8}$$

By the properties of norm and the orthogonality of the basis functions, we can simplify Equation (8) as follows:

$$\|f - f_n\| = \sqrt{\sum_{|\alpha|>n} |c_\alpha|^2 \|P_\alpha^d\|^2}. \tag{9}$$

Now, as n increases, the terms in the sum on the right-hand side become smaller because c_α are the coefficients of the original expansion (Equation (6)). Therefore, as n approaches infinity, $\|f - f_n\|$ approaches zero, which means that f_n converges to f in the norm of the given function space. Hence, f_n is a norm-attainable polynomial of degree n in the given function space, which completes the proof. □

Proposition 8. *Let $q_n(x)n = 0^\infty$ be a sequence of norm-attainable polynomials on the interval $[-1, 1]$ such that $qn(x)$ converges uniformly to a continuous function $g(x)$ on $[-1, 1]$. Then, $g(x)$ is also norm-attainable on the same interval.*

Proof. Let $q_n(x)$ be a sequence of norm-attainable polynomials on the interval $[-1, 1]$ such that $q_n(x)$ converges uniformly to a continuous function $g(x)$ on $[-1, 1]$. We want to show that $g(x)$ is also norm-attainable on the same interval. By the uniform convergence of $q_n(x)$ to $g(x)$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\|q_n(x) - g(x)\|_\infty = \sup_{x\in[-1,1]} |q_n(x) - g(x)| < \frac{\varepsilon}{2}.$$

Since each $q_n(x)$ is norm-attainable, there exists $x_n \in [-1, 1]$ such that $\|q_n\|_\infty = |q_n(x_n)|$. Now, consider the sequence of values $\{q_n(x_n)\}$. Since $|q_n(x_n) - g(x_n)| < \frac{\varepsilon}{2}$ for all $n \geq N$, we have

$$\begin{aligned} |q_n(x_n) - g(x_n)| &\leq |q_n(x_n) - g(x_n)| + |q_n(x_n) - g(x)| + |g(x) - g(x_n)| \\ &\leq |q_n(x_n) - g(x_n)| + \frac{\varepsilon}{2} + |g(x) - g(x_n)|. \end{aligned}$$

Now, using the triangle inequality, we have

$$|q_n(x_n) - g(x)| \leq |q_n(x_n) - g(x_n)| + |g(x) - g(x_n)| < \frac{\varepsilon}{2} + |g(x) - g(x_n)|.$$

Since $\lim_{n\to\infty} |g(x) - g(x_n)| = 0$, there exists N' such that for all $n \geq N'$, we have $|g(x) - g(x_n)| < \frac{\varepsilon}{2}$. Combining the results for $n \geq \max(N, N')$, we get

$$|q_n(x_n) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that the sequence $\{q_n(x_n)\}$ converges to $g(x)$, and since each $q_n(x_n)$ is a value of $g(x)$, we have shown that $g(x)$ is norm-attainable on the interval $[-1, 1]$. Therefore, the proposition is proved. □

Lemma 2. *Let $p_n(x)$ be a sequence of norm-attainable polynomials on the interval $[a, b]$, and let $q_n(x)n = 0^\infty$ be a sequence of polynomials such that $\lim n \to \infty q_n(x) = p_n(x)$ uniformly on $[a, b]$. If $q_n(x)$ is uniformly bounded on $[a, b]$, then $p_n(x)$ is also norm-attainable on the interval $[a, b]$.*

Proof. Let $\|q_n\|_\infty = \sup_{x\in[a,b]} |q_n(x)|$ be the supremum norm of $q_n(x)$ on $[a, b]$. Since $\{q_n(x)\}$ is uniformly bounded on $[a, b]$, there exists a constant $M > 0$ such that $\|q_n\|_\infty \leq M$ for all n. For each n, since $q_n(x)$ is a polynomial, it is continuous on the closed interval $[a, b]$. By the Weierstrass Approximation Theorem, there exists a sequence of polynomials $\{r_k(x)\}$ that converges uniformly to $q_n(x)$ on $[a, b]$. In other words, for each n, there exists a sequence of polynomials $\{r_{k_n}(x)\}$ such that

$$\lim_{k_n\to\infty} r_{k_n}(x) = q_n(x) \quad \text{uniformly on } [a, b].$$

Now, consider the sequence of polynomials $s_n(x) = r_{k_n}(x)$. By construction, we have

$$\lim_{n\to\infty} s_n(x) = \lim_{k_n\to\infty} r_{k_n}(x) = q_n(x) \quad \text{uniformly on } [a, b].$$

Since each $s_n(x)$ is a polynomial, we can apply the principle of uniform limit to conclude that $\lim_{n\to\infty} s_n(x)$ is also a continuous function on $[a, b]$. Now, let's show that $\lim_{n\to\infty} s_n(x)$ attains its norm on $[a, b]$. Since $s_n(x)$ converges uniformly to $q_n(x)$, for each $\varepsilon > 0$, there exists an N such that for all $n > N$ and for all $x \in [a, b]$, we have

$$|s_n(x) - q_n(x)| < \frac{\varepsilon}{2}.$$

Since $\|q_n\|_\infty \leq M$, we have $|q_n(x)| \leq M$ for all $x \in [a, b]$. Therefore, for all $n > N$ and for all $x \in [a, b]$, we have

$$|s_n(x)| \leq |s_n(x) - q_n(x)| + |q_n(x)| < \frac{\varepsilon}{2} + M.$$

Thus, $s_n(x)$ is bounded on $[a, b]$ for all $n > N$. Since the limit function $\lim_{n\to\infty} s_n(x)$ is continuous on $[a, b]$ and bounded, it attains its norm on $[a, b]$. Therefore, $p_n(x) = \lim_{n\to\infty} s_n(x)$ is also norm-attainable on the interval $[a, b]$. Thus, we have shown that if $\{q_n(x)\}$ is uniformly bounded and converges uniformly to $p_n(x)$ on $[a, b]$, then $p_n(x)$ is norm-attainable on $[a, b]$. □

Proposition 9. *Consider a family of orthogonal polynomials $P_n(x)n = 0^\infty$ defined on an interval $[a, b]$ with respect to a weight function $w(x)$. If $P_n(x)n = 0^\infty$ forms a complete orthogonal basis for a function space on $[a, b]$, and if a continuous function $f(x)$ can be expressed as a series expansion in terms of these polynomials, then $f(x)$ is norm-attainable on the interval $[a, b]$ with respect to the weight function $w(x)$.*

Proof. Let $f(x)$ be a continuous function on $[a, b]$ that can be expressed as a series expansion in terms of the orthogonal polynomials $\{P_n(x)\}_{n=0}^\infty$ as follows:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x).$$

We want to show that $f(x)$ is norm-attainable on the interval $[a, b]$ with respect to the weight function $w(x)$. To do this, we will construct a sequence of functions $\{f_k(x)\}_{k=1}^\infty$ in the function space that converges to $f(x)$ in the norm induced by the weight function

$w(x)$. For each k, define the function $f_k(x)$ as follows:

$$f_k(x) = \sum_{n=0}^{k} c_n P_n(x).$$

Now, we need to show that $\lim_{k\to\infty} \|f - f_k\|_w = 0$, where $\|\cdot\|_w$ is the norm induced by the weight function $w(x)$. Using the completeness of the orthogonal basis $\{P_n(x)\}_{n=0}^{\infty}$, we can write the difference between $f(x)$ and $f_k(x)$ as follows:

$$\begin{aligned} f(x) - f_k(x) &= \sum_{n=k+1}^{\infty} c_n P_n(x) \\ &= \sum_{n=k+1}^{\infty} \frac{\langle f - P_k, P_n\rangle_w}{\|P_n\|_w} P_n(x), \end{aligned}$$

where $\langle \cdot, \cdot \rangle_w$ denotes the inner product induced by the weight function $w(x)$. Now, we can estimate the norm of the difference using Cauchy-Schwarz inequality:

$$\begin{aligned} \|f - f_k\|_w^2 &= \int_a^b |f(x) - f_k(x)|^2 w(x)dx \\ &= \int_a^b \left| \sum_{n=k+1}^{\infty} \frac{\langle f - P_k, P_n\rangle_w}{\|P_n\|_w} P_n(x) \right|^2 w(x)dx \\ &\le \int_a^b \left(\sum_{n=k+1}^{\infty} \left| \frac{\langle f - P_k, P_n\rangle_w}{\|P_n\|_w} \right| |P_n(x)| \right)^2 w(x)dx \\ &\le \sum_{n=k+1}^{\infty} \frac{|\langle f - P_k, P_n\rangle_w|^2}{\|P_n\|_w^2} \int_a^b |P_n(x)|^2 w(x)dx. \end{aligned}$$

Now, since the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are orthogonal with respect to the weight function $w(x)$, we have $\langle P_m, P_n\rangle_w = 0$ for $m \neq n$, and $\|P_n\|_w^2 = \langle P_n, P_n\rangle_w$. Therefore, the above inequality simplifies to:

$$\|f - f_k\|_w^2 \le \sum_{n=k+1}^{\infty} \frac{|\langle f - P_k, P_n\rangle_w|^2}{\langle P_n, P_n\rangle_w}.$$

Now, we can use the properties of the inner product to further simplify:

$$\|f - f_k\|_w^2 \le \sum_{n=k+1}^{\infty} \frac{|\langle f - P_k, P_n\rangle_w|^2}{\langle P_n, P_n\rangle_w} = \sum_{n=k+1}^{\infty} |c_n|^2.$$

Since the series $\sum_{n=k+1}^{\infty} |c_n|^2$ converges to zero as k approaches infinity (because $f(x)$ is expressible as a series expansion), we have shown that $\lim_{k\to\infty} \|f - f_k\|_w = 0$. Therefore, $f(x)$ is norm-attainable on the interval $[a, b]$ with respect to the weight function $w(x)$, as the sequence of functions $\{f_k(x)\}_{k=1}^{\infty}$ converges to $f(x)$ in the norm induced by $w(x)$. $\square$

4. Conclusion

In this paper, we have investigated the norm-attainability of classical orthogonal polynomials, including Chebyshev, Hermite, Laguerre, and Legendre polynomials, within specific weight functions and intervals. We have established the conditions under which these polynomials can achieve norm-attainment in their respective Hilbert spaces. The study demonstrated the norm-attainability of Chebyshev polynomials under the weight function $(1-x^2)^{-1/2}$ on the interval $[-1, 1]$, proved the norm-attainment of Hermite polynomials under a normal distribution weight function, established the norm-attainability of Laguerre polynomials with a gamma distribution weight function on the positive real line, and verified the norm-attainment of Legendre polynomials with a weight function equal to 1 on the interval $[-1, 1]$.

References

[1] S. T. Ali and F. Bagarello, D-pseudo-bosons, complex Hermite polynomials, and integral quantization. Symmetry, Integr. Geom.: Meth. Appl. 11 (2015), 80-93.

[2] T. K. Araaya, Three systems of orthogonal polynomials and associated operators. Dissertation, Uppsala University, 2011.

[3] E. Asplund, Orthogonal polynomials in Hilbert space. Trans. Amer. Math. Soc., 144 (1969), 235-240.

[4] G. Bachman and L. Narici, Functional Analysis, Academic press, New York, 2000.

[5] V. S. Balaganski, Approximation properties of sets in Hilbert space, Mat. Zametki, 31 (1982), 785-800.

[6] J. M. Borwein, Orthogonal polynomials, Americam Mathematical Society, 2019.

[7] A. Fernando and J. K. Nigel, Topics in Banach space theory, Graduate Texts in Mathematics. Springer-Verlag, New York, 2006.

[8] D. Frank, Best Approximation in Inner Product Spaces, Springer, New York, 2001.

[9] P. R. Halmos , A Hilbert space Problem book, Springer-Verlag, New York, 1982.

[10] H. Mazaheri, T. D. Narang and H. R. Khademzadeh, Nearest and Farthest points in normed spaces, Yazd University, 2015.

[11] N. B. Okelo, On certain conditions for convex optimization in hilbert spaces, Khayyam J. Math. 5(2019), 108-112.

[12] N. B. Okelo, The norm-attainability of some elementary operators, Appl. Math. E-Notes, 13 (2013), 1-7.

[13] N. B. Okelo, α-Supraposinormality of operators in dense norm-attainable classes, Univ. J. Math. Appl. 2 (2019), 42-43.

[14] N. B. Okelo, On orthogonality of elementary operators in norm-attainable classes, Taiwan. J. Math. 24 (2020), 119-130.

[15] N. B. Okelo, J. O. Agure and P. O. Oleche, Various notions of orthogonality in normed spaces, Acta Math. Sci. 33 (2013), 1387-1397.

[16] T. J. Rivlin, An introduction to the approximation of functions, Dover Publications Inc., New York, 1981.

[17] Sangeeta and T. D. Narang, On the farthest points in convex metric spaces and linear metric spaces, Pub. l'Inst. Math. 95 (2014), 229-238.

[18] G. Szego, Orthogonal polynomials, American Mathematical Society, 1975.

[19] H. Weiner, Spectral theory of operators in Hilbert spaces. Dover Publications Inc., New York, 1992.

[20] L. P. Vlasov, Approximative properties of sets in normed linear spaces, Russ. Math. Surv. 28 (1973), 1-66.

[21] C. Zalinescu, Spectral Analysis in General Vector Spaces, World Scientific Press, 2002.