Decomposable Positive Map From $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_2(\mathcal{M}_2(\mathbb{C}))$

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ABSTRACT. In most literature, the decomposition of positive maps from $\mathcal{M}_3$ to $\mathcal{M}_2$ are discussed where the matrix elements are complex numbers. In this paper we construct a positive maps $\Phi_{(\mu,c_1,c_2)}$ from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_2(\mathcal{M}_2(\mathbb{C}))$. The Choi matrices for complete positivity and complete copositivity ares visualized as tensor matrix $\mathcal{M}_3 \otimes \mathcal{M}_2$ with $\mathcal{M}_2(\mathbb{C})$ as the entry elements. The construction allow us describe decomposability on positive semidefinite matrices.

1. INTRODUCTION

Positive linear maps on C^*-algebras particularly those of finite dimensions have been very important in quantum information theory and quantum channels. Stinespring [6] initiated the concept of completely positive maps with his representation(or dilation) theorem. Arveson in [1] and [2] found the application of completely positive maps in operator theory and further developed extensively in operator algebra and mathematical physics. Woronowicz [11], Theorem 3.1.6 showed that every positive linear map Φ form $\mathcal{M}_2(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is decomposable if and only if $m \leq 3$. In [7], [8] and [9] Theorem 1, Størmer gives conditions for decomposability of positive maps; For $\mathcal{A}$ be a C^*-algebra and linear map Φ is decomposable if and only if for all $n \in \mathbb{N}$ whenever (x_{ij}) and (x_{ji}) belong to $\mathcal{M}_n(\mathcal{A})^+$. Choi [4] gave the first example of indecomposable map, for a 3-dimension case.

Yang, Leung and Tang [12] showed that every 2-positive linear map from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_3(\mathbb{C})$ is decomposable. Though we are motivated by the question in [12] that enquire if there exist indecomposable 2-positive maps from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_4(\mathbb{C})$), we show there is a decomposable positive map from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_2(\mathcal{M}_2(\mathbb{C}))$.

In most literature, the authors have studied case $\mathcal{M}_3$ to $\mathcal{M}_2$ where the matrix elements are complex numbers. In this paper we construct a positive maps $\Phi_{(\mu,c_1,c_2)}$ from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_2$ where the matrix elements of $\mathcal{M}_2$ is a 2×2 positive matrix $\mathcal{M}_2(\mathbb{C})$. We find conditions on the triplet μ, c_1, c_2 for which the map is positive, completely positive, 2-positive and decomposable.

A matrix $X \in \mathcal{M}_n(\mathbb{C})$ is called positive semi-definite if it is hermitian and all its eigenvalues are positive. It is denoted as $X \geq 0$. The set of all positive semi-definite matrices in $\mathcal{M}_n(\mathbb{C})$ is denoted by $\mathcal{M}_n(\mathbb{C})^+$. Let the identity map on and the transpose map on $\mathcal{M}_n(\mathbb{C})^+$ be denoted by $\mathcal{I}_n$ and τ_n respectively. A linear map Φ is from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is called positive if $\Phi(\mathcal{M}_n(\mathbb{C}))^+ \subseteq \mathcal{M}_m(\mathbb{C})^+$. A map Φ from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is $k-positive$ if $\mathcal{I} \otimes \Phi : \mathcal{M}_k \otimes \mathcal{M}_n \longrightarrow \mathcal{M}_k \otimes \mathcal{M}_m$ is positive. On the other hand, Φ from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is $k-copositive$ if the map $\tau_m \otimes \Phi : \mathcal{M}_k \otimes \mathcal{M}_n \longrightarrow \mathcal{M}_k \otimes \mathcal{M}_n$ is positive. The Choi result in [3] affirms that the positive map Φ is completely positive if and only if it's Choi matrix is positive semidefinite.

2. POSITIVITY

Let $X \in \mathcal{M}_n(\mathbb{C})$ be a positive semidefinite matrix denoted by $X = [x_i \bar{x}_j]$, where $x_i = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ is a column vector and $\bar{x}_j$ is the transpose conjugate(row vector) of x_i. We denote the diagonal entries $x_n \bar{x}_n$ by α_n.

Let $X \in \mathcal{M}_3$ be a positive semidefinite matrix with complex entries. Let $0 < \mu \leq 1$, $c_1, c_2 > 0$ and $r \in \mathbb{N}$. Then we define the family of positive maps $\Phi_{(\mu, c_1, c_2)}$ as follows:

$$\Phi_{(\mu, c_1, c_2)} : \mathcal{M}_3(\mathbb{C})^+ \longrightarrow \mathcal{M}_2(\mathcal{M}_2)\mathbb{C})^+.$$

$$X \mapsto \left(\begin{array}{cc|cc} P_1^\mu & -c_1 x_1 \bar{x}_2 & 0 & -\mu x_1 \bar{x}_3 \\ -c_1 x_2 \bar{x}_1 & P_2^\mu & -c_2 x_2 \bar{x}_3 & 0 \\ \hline 0 & -c_2 x_3 \bar{x}_2 & P_3^\mu & 0 \\ -\mu x_3 \bar{x}_1 & 0 & 0 & P_4^\mu \end{array} \right), \tag{2.0.1}$$

where

$$\begin{aligned} P_1^\mu &= \mu^{-r}(\alpha_1 + c_1 \alpha_2 \mu^r + c_2 \alpha_3 \mu^r) \\ P_2^\mu &= \mu^{-r}(\alpha_2 + c_1 \alpha_3 \mu^r + c_2 \alpha_1 \mu^r) \\ P_3^\mu &= \mu^{-r}(\alpha_1 + \alpha_2 + \alpha_3) \\ P_4^\mu &= \mu^{-r}(\alpha_3 + c_1 \alpha_1 \mu^r + c_2 \alpha_2 \mu^r) \end{aligned}$$

The matrix $\Phi_{(\mu, c_1, c_2)}(X)$ is visualized as a 2×2 block matrix in $\mathcal{M}_2(\mathcal{M}_2(\mathbb{C}))$.

The linear map Φ is uniquely determined by the polynomial function:

$$F(z, x) := v\Phi(x_i \bar{x}_j) v^T$$

as a biquadratic function in $x := (x_1, x_2, x_3)$ and $v := (v_1, v_2, v_3, v_4)$. The map Φ is positive if and only if the biquadratic form $F(z, x)$ is a sum of squares (positive semi-definite).

We characterize of the positivity of the map Φ for $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ and $t \in \mathbb{C}$.

Lemma 2.1. *Let $0 < \mu < 1$ and $c_1, c_2 \geq 0$. Then the function*

$$\begin{aligned} F(v_1, v_2, v_3, v_4, t) &= \mu^{-r}(1 + c_1 \mu^r + c_2 |t| \mu^r) v_1^2 + \mu^{-r}(1 + c_1 |t| \mu^r + c_2 \mu^r) v_2^2 + \mu^{-r}(2 + |t|) v_3^2 \\ &\quad + \mu^{-r}(|t| + c_1 \mu^r + c_2 \mu^r) v_4^2 - 2c_1 v_1 v_2 - 2c_2 \Re(t) v_2 v_3 - 2\mu \Re(t) v_1 v_4 \end{aligned}$$

is positive semidefinite for every $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $t \in \mathbb{C}$ if and only if the following two conditions are satisfied:

$$\mu^{-r} > c_1. \tag{2.0.2}$$

$$\mu^{-r} > c_2. \tag{2.0.3}$$

Proof. If $v_1 = 0$. Then,
$F(0, v_2, v_3, v_4, t)$

$$
\begin{aligned}
&= \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 + \mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 - 2c_2\Re(t)v_2v_3 \\
&= (c_1|t| + c_2)v_2^2 + 2\mu^{-r}v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 + (\mu^{-r}v_2^2 - 2v_2v_3c_2\Re(t) + \mu^{-r}|t|v_3^2) \\
&= (c_1|t| + c_2)v_2^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 + \mu^{-r}(v_2 - \mu^r c_2\Re(t)v_3)^2 + (2\mu^{-r} + \mu^{-r}|t| - \mu^r c_2^2\Re(t)^2)v_3^2.
\end{aligned}
$$

$F(0, v_2, v_3, v_4, t)$ is positive when the coefficient of v_3^2 satisfy the inequality,

$$\mu^{-2r}(2 + |t|) - c_2^2\Re^2(t)^2 \geq 0. \tag{2.0.4}$$

Letting $t = x + iy$. We have that,

$$
\begin{aligned}
\mu^{-r}(2 + |t|) - \mu^r c_2^2\Re^2(t)^2 &= 2\mu^{-2r} + (\mu^{-2r}(|x|^2 + |y|^2) - x^2 c_2^2) \\
&= 2\mu^{-2r} + \mu^{-2r}|y|^2 + |x|^2(\mu^{-2r} - c_2^2)
\end{aligned}
$$

is positive whenever $\mu^{-r} \geq c_2$ hold.

If $v_2 = 0$. Then ,
$F(v_1, 0, v_3, v_4, t)$

$$
\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 - 2\mu\Re(t)v_1v_4 \\
&= \mu^{-r}(c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(c_1\mu^r + c_2\mu^r)v_4^2 + (\mu^{-r}v_1^2 - 2v_1v_4\mu\Re(t) + \mu^{-r}|t|v_4^2) \\
&= \mu^{-r}(c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(c_1\mu^r + c_2\mu^r)v_4^2 + \mu^{-r}(v_1 - \mu^{1+r}\Re(t)v_4)^2 + \mu^{-r}(|t| \\
&\quad - \mu^{2+2r}\Re(t)^2)v_4^2 \\
&\geq 0.
\end{aligned}
$$

If $v_3 = 0$. Then ,
$F(v_1, v_2, 0, v_4, t)$

$$
\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 - 2c_1v_1v_2 \\
&\quad - 2\mu\Re(t)v_1v_4 \\
&= c_2|t|v_1^2 + \mu^{-r}(1 + c_2\mu^r)v_2^2 + (c_1 + c_2)v_4^2 + (\mu^{-r}v_1^2 - 2v_1v_4\mu\Re(t) + \mu^{-r}|t|v_4^2) + c_1(v_1^2 - 2v_1v_2 + |t|v_2^2) \\
&= c_2|t|v_1^2 + \mu^{-r}(1 + c_2\mu^r)v_2^2 + (c_1 + c_2)v_4^2 + \mu^{-r}(v_1 - \mu^{1+r}\Re(t)v_4)^2 \\
&\quad + \mu^{-r}(|t| - \mu^{2+2r}\Re(t)^2)v_4^2 + c_1(v_1 - v_2)^2 + c_1(|t|^2 - 1)v_2^2
\end{aligned}
$$

$F(v_1, v_2, 0, v_4, t)$ is positive whenever $\mu^{-r} - c_1 \geq 0$ hold. That is, the coefficients of v_2^2 is such that,

$$\mu^{-r} + c_2 + c_1(|t| - 1) = (\mu^{-r} - c_1) + c_2 + c_1|t| \geq 0. \tag{2.0.5}$$

If $v_4 = 0$. Then,

$F(v_1, v_2, v_3, 0, t)$

$$
\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 + \mu^{-r}(2 + |t|)v_3^2 - 2c_1v_1v_2 - 2c_2\Re(t)v_2v_3 \\
&= \mu^{-r}(1 + c_2|t|\mu^r)v_1^2 + c_2v_2^2 + 2\mu^{-r}v_3^2 + c_1(v_1^2 - 2v_1v_2 + |t|v_2^2) + (\mu^{-r}v_2^2 - 2c_2\Re(t)v_2v_3 + \mu^{-r}|t|v_3^2) \\
&= \mu^{-r}(1 + c_2|t|\mu^r)v_1^2 + c_2v_2^2 + 2\mu^{-r}v_3^2 + c_1(v_1 - v_2)^2 + c_1(|t|^2 - 1)v_2^2 + \mu^{-r}(v_2 - \mu^r c_2\Re(t)v_3)^2 \\
&\quad + (\mu^{-r}|t| - \mu^r c_2^2\Re(t)^2)v_3^2 \\
&\geq \quad 0
\end{aligned}
$$

whenever the inequalities (2.0.4) and (2.0.5) hold.

Now let $v_i \neq 0, i = 1, 2, 3, 4$ and assume that there exist $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $t \in \mathbb{C}$ such that $v_1 \neq 0$ and $F(v_1, v_2, v_3, v_4, t) < 0$. Since $0 < \mu < 1$ and $c_1, c_2 \geq 0$. Then,

$F(v_1, v_2, v_3, v_4, t)$

$$
\begin{aligned}
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 + \mu^{-r}(2 + |t|)v_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 \\
&\quad - 2c_1v_1v_2 - 2c_2\Re(t)v_2v_3 - 2\mu\Re(t)v_1v_4 \\
&= \mu^{-r}v_1^2 + \mu^{-r}v_2^2 + 2\mu^{-r}v_3^2 + (c_1 + c_2)\mu^{-r}v_4^2 + c_1(v_1 - v_2)^2 + c_1(|t|^2 - 1)v_2^2 + \mu^{-r}(v_2 - \mu^r c_2\Re(t)v_3)^2 \\
&\quad + (\mu^{-r}|t| - \mu^r c_2^2\Re(t)^2)v_3^2 + \mu^{-r}(v_1 - \mu^{1+r}\Re(t)v_4)^2 + (\mu^{-r}|t| - \mu^{2+2r}\Re(t)^2)v_4^2 \\
&< \quad 0
\end{aligned}
$$

is a contradiction when the inequalities (2.0.4) and (2.0.5) hold . Thus $F(v_1, v_2, v_3, v_4, t) \geq 0$ for every $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $t \in \mathbb{C}$ $\qquad\square$

Proposition 2.2. *The linear map $\Phi_{(\mu,c_1,c_2)}$ is positive provided Lemma 2.1 is satisfied. are satisfied.*

Proof. We need to show that,

$$
\Phi\left(\begin{pmatrix} q \\ s \\ t \end{pmatrix} \begin{pmatrix} \bar{q} & \bar{s} & \bar{t} \end{pmatrix} \right) \in \mathcal{M}_4^+
$$

for every $q, s, t \in \mathbb{C}$.

Thai is.

$$
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}^T
\left(\begin{array}{cc|cc}
P_1^\mu & -c_1 q\bar{s} & 0 & -\mu q\bar{t} \\
-c_1 s\bar{q} & P_2^\mu & -c_2 s\bar{t} & 0 \\
\hline
0 & -c_2 t\bar{s} & P_3^\mu & 0 \\
-\mu t\bar{q} & 0 & 0 & P_4^\mu
\end{array} \right)
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \geq 0 \qquad (2.0.6)
$$

where,

$$
\begin{aligned}
P_1^\mu &= \mu^{-r}(|q|^2 + c_1|s|^2\mu^r + c_2|t|\mu^r) \\
P_2^\mu &= \mu^{-r}(|s|^2 + c_1|t|\mu^r + c_2|q|^2\mu^r) \\
P_3^\mu &= \mu^{-r}(|q|^2 + |s|^2 + |t|) \\
P_4^\mu &= \mu^{-r}(|t| + c_1|q|^2\mu^r + c_2|s|^2\mu^r)
\end{aligned}
$$

for every $v_1, v_2, v_3, v_4 \in \mathbb{R}$ and $q, s, t \in \mathbb{C}$.

Taking $q = s = 0$.

$$
F(v_1, v_2, v_3, v_4, t) = c_1\mu^{-r}|t|v_1^2 + c_1|t|v_2^2 + \mu^{-r}|t|v_3^2 + \mu^{-r}|t|v_4^2 \geq 0.
$$

If $q = 0$. Since $0 < \mu \leq 1$, by inequality (2.0.4),
$F(v_1, v_2, v_3, v_4, t)$

$$
\begin{aligned}
&= (c_1 + c_2|t|)v_1^2 + \mu^{-r}(1 + c_1|t|)v_2^2 + \mu^{-r}(1 + |t|)v_3^2 + (\mu^{-r}|t| + c_2)v_4^2 - 2c_2\Re(t)v_2v_3 \\
&\geq 0.
\end{aligned}
$$

If $s = 0$,
$F(v_1, v_2, v_3, v_4, t)$

$$
\begin{aligned}
&= \mu^{-r}(1 + c_2|t|)v_1^2 + (c_1|t| + c_2)v_2^2 + \mu^{-r}(1 + |t|)v_3^2 + \mu^{-r}(|t| + c_1\mu^r)v_4^2 - 2\mu\Re(t)v_1v_4 \\
&= c_2|t|v_1^2 + (c_1|t| + c_2)v_2^2 + \mu^{-r}(1 + |t|)v_3^2 + c_1v_4^2 + \mu^{-r}(v_1 - \mu^{1+r}\Re(t)v_4)^2 \\
&\quad + (\mu^{-r}|t| - \mu^{2+r}\Re(t)^2)v_4^2 \\
&\geq 0.
\end{aligned}
$$

If q and s are not equal to zero. Assume that $q = s = 1$. Then, by Lemma 2.1

$$
z^T \Phi \left(\begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix} \begin{pmatrix} 1 & 1 & \bar{t} \end{pmatrix} \right) z \geq 0
$$

since the polynomial,

$$
\begin{aligned}
&F(v_1, v_2, v_3, v_4, t) \\
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)v_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)v_2^2 + \mu^{-r}(2 + |t|)v_3^2 \\
&\quad + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)v_4^2 - 2c_1v_1v_2 - 2c_2\Re(t)v_2v_3 - 2\mu\Re(t)v_1v_4 \\
&= \mu^{-r}v_1^2 + \mu^{-r}v_2^2 + c_1(v_1 - v_2)^2 + c_1(|t|^2 - 1)v_2^2 + (2\mu^{-r} + \mu^{-r}|t| - \mu^r c_2^2\Re(t)^2)v_3^2 \\
&\quad + \mu^{-r}(v_2 - \mu^r c_2\Re(t)v_3)^2 + \mu^{-r}(v_1 - \mu^{1+r}\Re(t)v_4)^2 + (\mu^{-r}(c_1 + c_2 + |t|) \\
&\quad - \mu^{2+2r}\Re(t)^2)v_4^2 \\
&\geq 0
\end{aligned}
$$

for every $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ and $t \in \mathbb{C}$ $\square$

3. COMPLETE (CO)POSITIVITY

The structure of the Choi matrix $C_{\Phi_{(\mu, c_1, c_2)}} \in \mathcal{M}_3(\mathcal{M}_2(\mathcal{M}_2))$ is visualized as a block matrix whose entries are 2×2 matrices within the 6×6 matrix.

Proposition 3.1. *Let $\psi_{(\mu, c_1, c_2)}$ a positive map. The following are equivalent:*

    (i) $\Phi_{(\mu, c_1, c_2)}$ *is completely positive,*
    (ii) $\Phi_{(\mu, c_1, c_2)}$ *is 2-positive and,*
    (iii) $\mu^{-2r} > c_1^2 + c_2^2$.

Proof. $(ii) \Rightarrow (iii)$.

Assume $\Phi_{(\mu,c_1,c_2)}$ is 2-positive. Then,.

$$\mathcal{I}_2 \otimes \Phi_{(\mu,c_1,c_2)}(X) = \left(\begin{array}{cccc|cccc}
\mu^{-r} & . & . & . & . & -c_1 & . & -\mu \\
. & c_2 & . & . & . & . & . & . \\
\hline
. & . & \mu^{-r} & . & . & . & . & . \\
. & . & . & c_1 & . & . & . & . \\
\hline
. & . & . & . & c_1 & . & . & . \\
-c_1 & . & . & . & . & \mu^{-r} & -c_2 & . \\
\hline
. & . & . & . & . & -c_2 & \mu^{-r} & . \\
-\mu & . & . & . & . & . & . & \mu^{-r}
\end{array}\right) \tag{3.0.1}$$

in $\mathcal{M}_2(\mathcal{M}_2 \otimes \mathcal{M}_2)$ is positive semidefinite. Therefore,

$$\begin{vmatrix}
\mu^{-r} & -c_1 & 0 & -\mu \\
-c_1 & \mu^{-r} & -c_2 & 0 \\
0 & c_2 & \mu^{-r} & 0 \\
-\mu & 0 & 0 & \mu^{-r}
\end{vmatrix} \geq 0 \tag{3.0.2}$$

By calculation of minors, $\mu^{-r} > c_1$ and $\mu^{-2r} > c_1^2 + c_2^2$ are necessary conditions for 2-positivity of $\Phi_{(\mu,c_1,c_2)}$.

$(iii) \Rightarrow (i)$.

The Choi matrix is

$$C_{\Phi_{(\mu,c_1,c_2)}} = \left(\begin{array}{cccc|cccc|cccc}
\mu^{-r} & . & . & . & . & -c_1 & . & . & . & . & . & -\mu \\
. & c_2 & . & . & . & . & . & . & . & . & . & . \\
. & . & \mu^{-r} & . & . & . & . & . & . & . & . & . \\
. & . & . & c_1 & . & . & . & . & . & . & . & . \\
\hline
. & . & . & . & c_1 & . & . & . & . & . & . & . \\
-c_1 & . & . & . & . & \mu^{-r} & . & . & . & . & -c_2 & . \\
. & . & . & . & . & . & \mu^{-r} & . & . & . & . & . \\
. & . & . & . & . & . & . & c_2 & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & c_2 & . & . & . \\
. & . & . & . & . & . & . & . & . & c_1 & . & . \\
. & . & . & . & . & -c_2 & . & . & . & . & \mu^{-r} & . \\
-\mu & . & . & . & . & . & . & . & . & . & . & \mu^{-r}
\end{array}\right) \tag{3.0.3}$$

in $\mathcal{M}_3(\mathcal{M}_2 \otimes \mathcal{M}_2)$.

Recall that complete positivity of $\Phi_{(\mu,c_1,c_2)}$ is equivalent to positive definiteness [3] of $C_{\Phi_{(\mu,c_1,c_2)}}$. Since (iii) is satisfied, the inequality (3.0.2) holds, and consequently $C_{\Phi_{(\mu,c_1,c_2)}}$ is positive definite. Hence, complete positivity of $\psi_{(\mu,c_1,c_2)}$ follows.

$(i) \Rightarrow (ii)$.

It follows from Proposition 3.1 that 2-positivity of $\Phi_{(\mu,c_1,c_2)}$ implies complete positivity.

$\square$

The Partial Positive transposition is operated with respect to the blocks $\mathcal{M}_2$ as the entry elements of the matrix $\mathcal{M}_3(\mathcal{M}_2)$. This leads to the Choi matrix $C^{\Gamma}_{\Phi_{(\mu,c_1,c_2)}} \in \mathcal{M}_3(\mathcal{M}_2)$ with the structure given in the next proposition.

Proposition 3.2. *Let $\psi_{((\mu,c_1,c_2)}$ be a positive map. Then following conditions are equivalent:*

(i) $\Phi_{(\mu,c_1,c_2)}$ *is completely copositive,*

(ii) $\Phi_{(\mu,c_1,c_2)}$ *is 2-copositive and,*

(iii) $c_1\mu^{-r} \geq c_2^2$

Proof. $(ii) \Rightarrow (iii)$.

Assume $\Phi_{(\mu,c_1,c_2)}$ is 2-copositive. Then,

$$\tau_2 \otimes \Phi_{(\mu,c_1,c_2)}(X) = \left(\begin{array}{cc|cc||cccc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot \\
\cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \mu^{-r} & \cdot & \cdot & -\mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & -c_2 \\
-c_1 & \cdot & -\mu & \cdot & \cdot & \mu^{-r} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \mu^{-r}
\end{array}\right) \tag{3.0.4}$$

in $\mathcal{M}_2(\mathcal{M}_2 \otimes \mathcal{M}_2)$ is positive semidefinite with the minors positive when conditions in (iii) holds.

$(iii) \Rightarrow (i)$

The choi matrix,

$$C^{\Gamma}_{\Phi_{(\mu,c_1,c_2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-c_1 & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & -c_2 & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & c_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & -\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_1 \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right)$$
$$\tag{3.0.5}$$

in $\mathcal{M}_3(\mathcal{M}_2 \otimes \mathcal{M}_3)$. Since (iii) is satisfied, by calculation of the minor, $C^{\Gamma}_{\Phi_{(\mu,c_1,c_2)}}$ is positive semidefinite when $\mu^{-r} \geq c_1$ holds. Hence, complete copositivity follows.

$(i) \Rightarrow (ii)$ follows from Proposition 2.2 that 2-positivity of $\Phi_{(\mu,c_1,c_2)}$ implies complete positivity. $\qquad\square$

Example 3.3. *When $r = 3, \mu = \frac{1}{2}, c_1 = 1$ and $c_2 = 2$. Then,*

$$
C_{\Phi_{(\frac{1}{2},1,2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
8 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & -2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 8 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8
\end{array}\right)
$$

with eigenvalues

$$\{10.2477, 8.4449, 8., 8., 7.5551, 5.75232, 2., 2., 2., 1., 1., 1.\}$$

and

$$
C^{\Gamma}_{\Phi_{(\frac{1}{2},1,2)}} = \left(\begin{array}{cc|cc||cc|cc||cc|cc}
8 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8
\end{array}\right)
$$

with eigenvalues

$$\{9., 8.03553, 8., 8., 8., 7., 4., 2., 1., 1., 0.964466, 0.\}$$

.

4. DECOMPOSABILITY OF $\Phi_{(\mu,c_1,c_2)}$

A positive linear map is decomposable if it is the sum of a completely positive linear map and a completely copositive linear map. The result of Choi [3] shows that a positive linear map Φ from $\mathcal{M}_n$ to $\mathcal{M}_m$ is decomposable if and only if there exist $n \times m$ matrices v_i and W_j such that,

$$\Phi(X) = V_i X V_i^* + W_j X^T W_j^*$$

for every X in $\mathcal{M}_n$, where T is the transpose of X.

Proposition 4.1. *The linear map $\Phi_{(\mu,c_1,c_2)}$ is decomposable.*

Proof. Let $\eta, \xi \in (0,1)$ and $a_i, b_i \in \mathbb{R}^+$ for $i = 1, 2$ such that $\eta^{-r} + \xi^{-r} = \mu^{-r}$ and $a_i + b_i = c_i$. We show that there exist 2-positive map $\Phi_{(\eta,a_1,a_2)}$ and 2-copositive map $\Phi_{(\xi,b_1,b_2)}$ whose sum is $\Phi_{(\mu,c_1,c_2)}$. Let $C_{\Phi_{(\mu,c_1,c_2)}}$ be

$$
\left(
\begin{array}{cccc|cccc|cccc}
\eta^{-r}+\xi^{-r} & \cdot & \cdot & \cdot & \cdot & -(a_1+b_1) & \cdot & \cdot & \cdot & \cdot & \cdot & -q\mu \\
\cdot & (a_2+b_2) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \eta^{-r}+\xi^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -(1-q)\mu & \cdot & \cdot \\
\cdot & \cdot & \cdot & (a_1+b_1) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & (a_1+b_1) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-(a_1+b_1) & \cdot & \cdot & \cdot & \cdot & \eta^{-r}+\xi^{-r} & \cdot & \cdot & \cdot & \cdot & -a_2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r}+\xi^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (a_2+b_2) & -b_2 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -b_2 & (a_2+b_2) & \cdot & \cdot & \cdot \\
\cdot & \cdot & -(1-q)\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (a_1+b_1). & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r}+\xi^{-r} & \cdot \\
-q\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r}+\xi^{-r}
\end{array}
\right)
\tag{4.0.1}
$$

in $\mathcal{M}_3(\mathcal{M}_2(\mathcal{M}_2)\mathbb{C})$ give be the sum of;

$$
C_{\Phi_{(\eta,a_1,a_2)}} =
\left(
\begin{array}{cccc|cccc|cccc}
\eta^{-r} & \cdot & \cdot & \cdot & \cdot & -a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & -q\mu \\
\cdot & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-a_1 & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & \cdot & -a_2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_2 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -a_2 & \cdot & \cdot & \cdot & \cdot & \eta^{-r} & \cdot \\
-q\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta^{-r}
\end{array}
\right)
$$

and

$$C_{\Phi_{(\xi,b_1,b_2)}} = \begin{pmatrix}
\xi^{-r} & . & . & . & . & -b_1 & . & . & . & . & . & . \\
. & b_2 & . & . & . & . & . & . & . & . & . & . \\
. & . & \xi^{-r} & . & . & . & . & . & . & -(1-q)\mu & . & . \\
. & . & . & b_1 & . & . & . & . & . & . & . & . \\
. & . & . & . & b_1 & . & . & . & . & . & . & . \\
-b_1 & . & . & . & . & \xi^{-r} & . & . & . & . & . & . \\
. & . & . & . & . & . & \xi^{-r} & . & . & . & . & . \\
. & . & . & . & . & . & . & b_2 & -b_2 & . & . & . \\
. & . & . & . & . & . & . & -b_2 & b_2 & . & . & . \\
. & . & -(1-q)\mu & . & . & . & . & . & b_1 . & . & . \\
. & . & . & . & . & . & . & . & . & \xi^{-r} & . \\
. & . & . & . & . & . & . & . & . & . & \xi^{-r}
\end{pmatrix}$$

When $q = 1$. Then, from the Choi matrices $C_{\Phi_{(\eta,a_1,a_2)}}$ and $C_{\Phi_{(\xi,b_1,b_2)}}$ the linear maps $\Phi(\eta, a_1, a_2)$ is completely positive and $\Phi(\xi, b_1, b_2)$ is completely copositive. On the other hand, when $q = 0$. Then $\Phi(\eta, a_1, a))$ is completely copositive and $\Phi(\xi, b_1, b_2)$ is completely positive. Hence, $\Phi(\mu, c_1, c_2)$ is decomposable. $\qquad\square$

5. Conclusion

It is known that every positive linear map Φ form $\mathcal{M}_2(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is decomposable if and only if $m \leq 3$. The map $\Phi_{(\mu,c_1,c_2)}$ from $\mathcal{M}_3(\mathbb{C})$ to $\mathcal{M}_2(\mathcal{M}_2(\mathbb{C}))$ is also decomposable with 2×2 matrices as the entry elements of the Choi matrix in $\mathcal{M}_3(\mathcal{M}_2)(\mathbb{C})$. However, a look at the example by Woronowicz [11] and Tang' [10] of a map from $\mathcal{M}_2(\mathbb{C})$ to $\mathcal{M}_4(\mathbb{C})$ when approached as a map from $\mathcal{M}_2(\mathbb{C})$ to $\mathcal{M}_2(\mathcal{M}_2(\mathbb{C}))$ fails to be decomposable with 2×2 matrices as the elements of it's Choi matrix.Note that the decomposition of these maps is not unique.

Declaration of competing interest

There is no competing interest.

Acknowledgment

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