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Well-Posedness for Rational Contraction On b-Metric Spaces

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ABSTRACT. The purpose of the present paper is to prove a new fixed point theorem for self mapping satisfying a generalized contractive conditions on complete b-metric spaces. We also study the well-posedness of a fixed point problem.

1. Introduction and preliminaries

Metric fixed point theory is a very extensive area of analysis with various applications. It is well known that the Banach contraction principle (1922 [4]) is fundamental in fixed point theory, it guarantees the existence and uniqueness of a fixed point for certain self-mapping in metric spaces.
Another direction, the notion of b-metric space was introduced by Bakhtin [3] and Czerwik [8] by replacing the triangular inequality by a rectangular one, This notion was the starting point for developing the fixed point theory in b-metric space with a view of generalizing the Banach contraction mapping theorem. Recently, many interesting fixed point theorems are proved in the framework of b-metric spaces, (see [6–11, 13–15, 18, 19]) In the sequel, We give some definitions and results which will be useful in this paper.

Definition 1.1. *([3, 8]) Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}^+$ is called a b-metric with coefficient $s \geq 1$ if it satisfies the following properties for each $x, y, z \in X$*

(1) $d(x, y) = 0$ *if and only if* $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then the pair (X, d) is called a b-metric space.

Rermark 1.1. *Every metric space is b-metric space with $s = 1$, but b-metric space need not necessarily be a metric space and the class of b-metric spaces is larger than the class of metric spaces.*

The most interesting example of b-metric is the following.

Example 1.1. *([5, 12])*

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(1) *Let* $X := l_p(\mathbb{R})$ *with* $0 < p < 1$ *where* $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ *. Define* $d : X \times X \to \mathbb{R}^+$ *as:*

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}$$

where $x = x_n, y = y_n$*. Then* d *is a b-metric space with coefficient* $s = 2^{1/p}$*.*

(2) *Let* $L_p([0,1]) = \{f : [0,1] \longrightarrow \mathbb{R} \ : \ \|f\|_{L_p([0,1])} < \infty\}$*,* $(0 < p < 1)$ *and*

$$\|f\|_{L_p([0,1])} = \left(\int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Denote $X = L_p([0,1])$*, define a mapping* $d : X \times X \to \mathbb{R}^+$ *by*

$$d(x,y) = \left(\int_0^1 |f(x) - g(x)|^p dx \right)^{\frac{1}{p}}.$$

Then d *is a b-metric space with coefficient* $s = 2^{1/p}$*.*

Definition 1.2. *[13] Let* (X,d) *be a b-metric space and* $\{x_n\}$ *a sequence in* X*. We say that:*

(1) $\{x_n\}$ *converges to* x *if* $d(x_n, x) \to 0$*, as* $n \to +\infty$*,*

(2) $\{x_n\}$ *is Cauchy sequence if* $d(x_n, x_m) \to 0$*, as* $n, m \to +\infty$*,*

(3) (X,d) *is complete if every Cauchy sequence in* X *is convergent.*

Each convergent sequence in a b-metric space has a unique limit and it is also a Cauchy sequence. Moreover, in general, a b-metric is not necessarily continuous. The following example illustrates this claim.

Example 1.2. *[13] Let* $X = \mathbb{N} \cup \{\infty\}$*. We define a mapping* $d : X \times X \longrightarrow \mathbb{R}^+$ *as follows:*

$$d(m,n) = \begin{cases} 0 & \text{if } m = n \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } \ m,n \text{ is even and the other is even or } \ \infty \\ 5 & \text{if one of } \ m,n \text{ is odd and the other is odd or } \ \infty \\ 2 & \text{otherwise } m = n. \end{cases}$$

Then (X,d) *is a b-metric space with coefficient* $s = \frac{5}{2}$*. However, let* $x_n = 2n$ *for each* $n \in \mathbb{N}$*. Then* $\lim_{n\to\infty} d(2n, \infty) = \lim_{n\to\infty} \frac{1}{2n} = 0$*, that is,* $x_n \to \infty$*, but* $d(x_n, 1) = 2 \nrightarrow 5 = d(\infty, 1)$ *as* $n \to \infty$*.*

Proposition 1.1. *[6] In a b-metric space* (X,d)*, the following assertions hold:*

- *A convergent sequence has a unique limit;*
- *Each convergent sequence is Cauchy;*
- *A metric space* (X,d) *is complete if every Cauchy sequence is convergent in* X*.*

Definition 1.3. *[21] Let* (X,d) *be a metric space and* $T : X \to X$ *a mapping. The fixed point problem of* T *is said to be well-posed if*

(i) T *has a unique fixed point* z *in* X*.*

(*ii*) *For any sequence* $\{y_n\}$ *in* X *such that* $\lim_{n\to\infty} d(Ty_n, y_n) = 0$, *we have* $\lim_{n\to\infty} d(y_n, z) = 0$.

Definition 1.4. *[23] Let* (X, d) *be a b-metric space with the coefficient* $s \geq 1$ *and let* $T : X \to X$ *be a given mapping. We say that* T *is continuous at* $u \in X$ *if for every sequence* (x_n) *in* X, *we have* $x_n \to u$ *as* $n \to \infty$ *then* $Tx_n \to Tu$ *as* $n \to \infty$. *If* T *is continuous at each point* $u \in X$, *then we say that* T *is continuous on* X.

Definition 1.5. *[21] Let* (X, d) *be a metric space and* $T : X \to X$ *a mapping. The fixed point problem of* T *is said to be well-posed if*

i) T *has a unique fixed point* u *in* X.

ii) *for any sequence* $\{y_n\}$ *in* X *such that* $\lim_{n\to\infty} d(Ty_n, y_n) = 0$, *we have* $\lim_{n\to\infty} d(y_n, u) = 0$.

Lemma 1.1. *[20] Let* (X, d) *be a complete b-metric space and let* $\{x_n\}$ *be a sequence in* X *such that*

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}), \text{ for all } n = 0, 1, 2, \ldots$$

where $0 \leq \lambda < 1$. *Then* $\{x_n\}$ *is a Cauchy sequence in* X.

2. Mains results

Firstly, we have the following useful lemmas

Lemma 2.1. *Let* (X, d) *be a complete b-metric space and let* $\{x_n\}$ *be a sequence in* X *such that*

$$d(x_n, x_{n+1}) \leq \beta_n d(x_{n-1}, x_n), \text{ for all } n = 1, 2, 3, \ldots \tag{2.1}$$

where $0 \leq \beta_n = \dfrac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}$.

Then

(1) $\beta_n < 1$ *for all* $n = 1, 2, 3, \ldots$;

(2) $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ *for all* $n = 1, 2, 3, \ldots$;

(3) $\{x_n\}$ *is a Cauchy sequence in* X.

Proof. (1) Assume that $x_n \neq x_{n+1}$ for each $n \geq 1$, and let $d_{n-1} = d(x_{n-1}, x_n)$, can be written as

$$\beta_n = \frac{d_{n-1} + d_n}{d_{n-1} + d_n + 1},$$

we show that $\beta_n < 1$, for all $n > 0$.

Observe that

$$0 \leq d_{n-1} + d_n < d_{n-1} + d_n + 1,$$

we conclude that

$$\frac{d_{n-1} + d_n}{d_{n-1} + d_n + 1} = \beta_n < 1. \tag{2.2}$$

Using (2.1) and (2.2), it follows that

$$d_n \leq \beta_n d_{n-1} < d_{n-1}, \text{ for all } n > 0, \tag{2.3}$$

and also

$$d_{n-1} \leq \beta_{n-1} d_{n-2} < d_{n-2}, \text{ for all } n > 1. \tag{2.4}$$

From (2.3) and (2.4),we find that

$$d_n < d_{n-2}.$$

So, by the above inequality we get

$$0 < d_n + d_{n-1} < d_{n-1} + d_{n-2},$$

and

$$0 < d_n + d_{n-1} + 1 < d_{n-1} + d_{n-2} + 1,$$

hence

$$\frac{d_n + d_{n-1}}{d_n + d_{n-1} + 1} < \frac{d_{n-1} + d_{n-2}}{d_{n-1} + d_{n-2} + 1},$$

is equivalent to $\beta_n < \beta_{n-1}$, continuing this process, we get

$$\beta_n < \beta_{n-1} < \cdots < \beta_1.$$

Accordingly, by Lemma 2.1 with $\lambda = \beta_1 < 1$, then $\{x_n\}$ is a Cauchy sequence in X.

□

Now, we shall state and prove our main results

Theorem 2.1. *Let (X, d) be a complete b-metric space with a coefficient $s \geq 1$ and $T : X \to X$ be self maps satisfying, for all $x, y \in X$*

$$\begin{aligned} sd(Tx, Ty) \leq & \frac{d(x, Tx) + d(y, Ty)}{d(x, y) + d(y, Ty) + 1} \\ & \times \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}, \end{aligned} \tag{2.5}$$

Then, T has a unique fixed point in X and the fixed point problem of T is well-posed.

Proof. **Step 1 existences**
For any arbitrary point, $x_0 \in X$. Define the sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \text{ for all } n \in \mathbb{N}.$$

If $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}$, then $u = x_{n_0}$ forms a fixed point for T that the proof finishes. Consequently, from now on, we assume that

$$x_{n+1} \neq x_n$$

First, we will show that $\{x_n\}$ is a Cauchy sequence in X.
Employing the inequality (2.5), we have

$$\begin{aligned} &sd(x_n, x_{n+1}) \\ &= d(Tx_{n-1}, Tx_n) \\ &\leq \frac{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)}{d(x_{n-1}, x_n) + d(x_n, Tx_n) + 1} \\ &\qquad \times \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2s}\right\} \\ &= \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} \\ &\qquad \times \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2s}\right\} \\ &= \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} \\ &\qquad \times \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\right\} \\ &\leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} \\ &\qquad \times \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{s\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]}{2s}\right\} \\ &= s\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} d(x_{n-1}, x_n), \end{aligned} \tag{2.6}$$

then

$$d(x_n, x_{n+1}) \leq \frac{\beta_n}{s} d(x_{n-1}, x_n),$$

where

$$\beta_n = \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}.$$

Applying Lemma 2.1, we deduce that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists a point $u \in X$ such that $\lim_{n\to\infty} x_n = u$.
Next, we will prove that $Tu = u$.
Therefore, using (2.5), we have

$$\begin{aligned} &sd(Tx_n, Tu) \\ &\leq \frac{d(x_n, Tx_n) + d(u, Tu)}{d(x_n, u) + d(u, Tu) + 1} \max\left\{d(x_n, u), d(x_n, Tx_n), d(u, Tu), \frac{d(x_n, Tu) + d(u, Tx_n)}{2s}\right\}, \end{aligned}$$

Taking the limit as $n \to +\infty$, we obtain that

$$\begin{aligned} d(u, Tu) &\leq \frac{d(u, Tu)^2}{s\left[d(u, Tu) + 1\right]} \\ &< d(u, Tu). \end{aligned}$$

Therefore, u is a fixed point of T.

Step 2 uniqueness

For the uniqueness, we assume that $u \neq v$ is another fixed point of T.

From the inequality (2.5), we find

$$\begin{aligned} sd(u,v) &= sd(Tu,Tv) \\ &\leq \frac{d(u,Tu)+d(v,Tv)}{d(u,v)+d(v,Tv)+1} \max\left\{d(u,v), d(u,Tu), d(v,Tv), \frac{d(u,Tv)+d(v,Tu)}{2s}\right\} \\ &= 0. \end{aligned}$$

This means

$$d(u,v) = 0.$$

Step 3

Let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} d\left(Ty_n, y_n\right) = 0$. We have

$$d\left(y_n, u\right) \leq s\left[d\left(y_n, Ty_n\right) + d\left(Ty_n, Tu\right)\right]. \tag{2.7}$$

Using the inequality (2.5), we get

$$\begin{aligned} & sd\left(Ty_n, Tu\right) \\ & \leq \frac{d(y_n,Ty_n)+d(u,Tu)}{d(y_n,u)+d(u,Tu)+1} \times \max\left\{d(y_n,u), d(y_n,Ty_n), d(u,Tu), \frac{d(y_n,Tu)+d(u,Ty_n)}{2s}\right\} \\ & \leq \frac{d(y_n,Ty_n)+0}{d(y_n,u)+1} \\ & \quad \times \max\left\{d(y_n,u), d(y_n,Ty_n), 0, \frac{s\left[d(y_n,u)+d(u,Tu)\right]+s\left[d(u,Tu)+d(Tu,Ty_n)\right]}{2s}\right\}. \end{aligned}$$

Therefore

$$sd\left(Ty_n, Tu\right) \leq \frac{d(y_n,Ty_n)}{d(y_n,u)+1} \max\left\{d(y_n,u), d(y_n,Ty_n), \frac{d(y_n,u)+d(Tu,Ty_n)}{2}\right\}. \tag{2.8}$$

Then passing to the limit

$$s \lim_{n\to\infty} d\left(Ty_n, Tu\right) = 0,$$

from 2.7, we conclude

$$\lim_{n\to\infty} d\left(y_n, u\right) = 0.$$

That is the fixed point problem of T is well-posed. This completes the proof. □

The following corollaries can be deduced as particular cases of the main theorem.

Corollary 2.1. *Let (X,d) be a complete b-metric space with a coefficient $s \geq 1$ and $T : X \to X$ be self maps satisfying, for all $x, y \in X$*

$$sd(Tx,Ty) \leq \frac{d(x,Tx)+d(y,Ty)}{d(x,y)+d(y,Ty)+1} d(x,y), \tag{2.9}$$

Then, T has a unique fixed point in X and the fixed point problem of T is well-posed.

Proof. Take $\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\} = d(x,y)$ in Theorem 2.1. □

Corollary 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be self maps satisfying, for all $x, y \in X$*

$$\begin{aligned} d(Tx, Ty) \le & \frac{d(x, Tx) + d(y, Ty)}{d(x, y) + d(y, Ty) + 1} \\ & \times \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}, \end{aligned} \tag{2.10}$$

Then, T has a unique fixed point in X and the fixed point problem of T is well-posed

Proof. Take $s = 1$ in Theorem 2.1 □

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